h-p Spectral Element Methods for Elliptic Problems on Non-smooth Domains using Parallel Computers

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Certificate

It is certified that the work contained in the thesis entitled "***h-p* Spectral Element Methods for Elliptic Problems on Non-smooth Domains using Parallel Computers**", by "**Satyendra Kumar Tomar (Roll No.: 9510864)**" has been carried out under our supervision and that this work has not been submitted elsewhere for a degree.

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to

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Synopsis

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Thesis Title: ***h-p* Spectral Element Methods for Elliptic Problems on Non-smooth Domains using Parallel Computers**

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We present *h-p* type Spectral Element Methods for Elliptic Boundary Value Problems on Polygonal Domains using Parallel Computers. We first discuss the Poisson equation on a polygon.

To resolve the singularity which arises at the corners we use a *geometric mesh* as has been proposed by Babuska and Guo. With this mesh we seek a solution which minimizes a weighted L^2 norm of the residuals in the partial differential equation and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity by adding a term which measures the jump in the function and its derivatives at *inter-element* boundaries, in an appropriate fractional Sobolev norm, to the functional being minimized. Since the second derivatives of the actual solution are not square integrable in a neighborhood of the corners we have to multiply the residuals in the partial differential equation by an appropriate power of r_k, where r_k measures the distance between the point P and the vertex A_k in a sectoral neighborhood of each of these vertices. In a neighborhood of the corner A_k we switch to new variables (τ_k, θ_k)

where $\tau_k = \ln r_k$ and (r_k, θ_k) are polar coordinates with origin at A_k. In doing so the geometrical mesh is reduced to a *quasi-uniform mesh* in a sectoral neighborhood of the corners and so *Sobolev's embedding theorems* and the *trace theorems* for Sobolev spaces apply for functions defined on mesh elements in these new variables with a ***Uniform Constant***. We then derive *differentiability estimates* with respect to these new variables and a *stability estimate* for the functional we minimize.

To solve the minimization problem we have defined, we need to solve the *normal equations* for the *least-squares problem* corresponding to collocating the partial differential equation and the boundary conditions at an over-determined set of collocation points and enforcing continuity of the function and its derivatives at the collocation points at inter-element boundaries, suitably weighted. However, we do not need to compute and store any matrices, like the *mass and stiffness matrices*, to compute the residual in the normal equations.

We then use the stability estimates to obtain *parallel preconditioners* and *error estimates* for the solution of the *minimization problem.* We can precondition the normal equations by using this preconditioner which is of *block diagonal form* and which allows the solutions for different elements to *decouple* completely. Moreover this is nearly optimal as the condition number of the preconditioned system is *polylogarithmic* in N, which is proportional to the number of processors and the number of degrees of freedom in each variable on each element. Finally we show that the error we commit is *exponentially small* in N.

For the purely Dirichlet problem our *spectral element functions* are *non-conforming* and hence there are no *common boundary values* to solve for. This no longer holds for problems with Mixed boundary conditions. Here our spectral element functions are *essentially non-conforming* except that they are continuous at the vertices of the elements on which they are defined. Hence the set of common boundary values are the values of the function at the vertices of the elements and so their *cardinality*, being proportional to the number of elements, is *small* as compared to Finite Element

Methods, where the common boundary values are the values of the functions along the edges of their elements. Thus the method proposed is a *Vertex-Based Method.*

We solve the normal equations by the *preconditioned conjugate gradient method.* We first solve a much smaller system of equations corresponding to the *Schur complement* of the sub-vector of common boundary values for which we need to be able to compute a preconditioner. The Schur complement matrix can be computed accurately since it's dimension is small. This method turns out to be computationally more efficient than Finite Element Methods where complex techniques have to be used to obtain a preconditioner for the Schur complement matrix. Moreover since for the problem with Dirichlet boundary conditions the spectral element functions are non-conforming, there is no set of common boundary values and so the method is even more efficient.

We should mention that once we have obtained our approximate solution consisting of non-conforming spectral element functions we can make a correction to it so that the corrected solution is conforming and is an exponentially accurate approximation to the actual solution in the H^1 norm over the whole domain.

All these results are valid for elliptic problems with mixed boundary conditions on domains with *curvilinear boundaries* which satisfy the usual conditions.

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At the very outset I take this opportunity to express my profound sense of gratitude and deep regards to my thesis supervisors Dr. Pravir Dutt and Dr. B.V. Rathish Kumar for their valuable suggestions, constructive criticism, consistent encouragement, patience and interest in guiding me throughout the course of this research work. They not only helped me in my attempts to gain insight into the fields related to thesis work but also in my all round development for which, I will be forever indebted to both of them.

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30.01.01

Satyendra Kumar Tomar

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Chapter 1

Introduction

This dissertation is on ***h-p*** **Spectral Element Methods for Elliptic Problems on Non-smooth Domains using Parallel Computers**. We consider here the two dimensional problem.

1.1 Review of Existing Work

Spectral methods are one of the three major techniques for the numerical solution of Partial Differential Equations (PDE), namely Finite Difference Methods (FDM), Finite Elements Methods (FEM) and Spectral Methods (SM). Among these, Spectral methods have emerged as among the most accurate way of solving PDE and have become increasingly popular for problems where high accuracy is desired for complex problems like numerical simulations of turbulent flows, numerical weather prediction, ocean dynamics etc. If properly constructed these methods converge to the solution with *exponential accuracy* and can be used to obtain solutions where other numerical techniques fail. The very high accuracy of Spectral methods allows one to treat problems which would require an enormous number of mesh points by finite differences, with much fewer degrees of freedom. However, a poorly designed Spectral method may perform much worse than simpler FDM.

In 1970s Gottlieb, Orszag and others initiated the work on problems in fluid dynam-

ics using Spectral methods and presented the formulation of modern Spectral methods in the monograph [22]. The book of Canuto, Hussaini, Quarteroni and Zang [14] focuses on fluid dynamics algorithms and includes practical as well theoretical aspects of Spectral methods. Among other references are the book by Voigt, Gottlieb and Hussaini [45] and the latest one in this series is by Karniadakis and Sherwin [28].

Spectral methods are based on global interpolants, in contrast to FDM and FEM. The basic principle of this numerical technique is to represent the dependent variable in a finite (truncated) series of known infinitely differentiable global functions, for instance,

$$u\left(x\right) \simeq u_N\left(x\right) = \sum_{n=0}^{N} a_n \phi_n,$$

where $\phi_n\left(x\right)$ are the approximation functions (also known as the trial or expansion functions) and which are used as the basis functions for a truncated series expansion of the solution. The series is then substituted into the differential (or integral) equation and upon the minimization of the residual function the unknown coefficients are computed.

Most commonly used *approximation* functions are Trigonometric polynomials (for periodic boundary conditions), Chebyshev polynomials and Legendre polynomials. The three most commonly used methods are *Collocation*[1], *Galerkin* and *Spectral Tau*, which differ in their choice of *test* (*or weight*) functions. The *Collocation* approach is the simplest and most widely used among these methods. The most effective choice for the grid points are those which corresponds to quadrature formula of maximum precision, like *Legendre-Gauss-Lobatto* points.

1.2 Spectral Methods on Non-smooth Domains

Unlike FEM and FDM, the order of the convergence of Spectral methods is not fixed and it is related to the maximum regularity (smoothness) of the solution. The high accuracy of Spectral methods is easily lost if the solution has finite regularity (i.e. in the presence of discontinuities) or the domain is irregular.

The latest book on **Spectral/*hp* Element Methods for CFD** by Karniadakis and Sherwin [28] summarizes the recent research in the subject. Here we present a brief summary of the Section *Non-Smooth Domains* in the Chapter *Helmholtz Equation.*

Exponential convergence is obtained with spectral/*hp* element methods if the solution is smooth, possessing a high degree of regularity. However, there are a number of cases for which the solution of a Poisson's equation may be singular, like solutions in a non-smooth domains, smooth domains but with a discontinuity in the boundary conditions or in the specified data (eg. forcing). Here, as in [28], we also assume that all the data, as well as the boundary conditions, are smooth and that singularities are due to corners in the domain. First derivatives are unbounded when the angle is reflexive, and second derivative are unbounded when the angle is acute or obtuse. In this case not only the fast convergence of spectral/*hp* discretizations may by lost, but also the numerical solution obtained (with any standard method) may be erroneous.

There are four methods discussed in that section, that allow us to recover, *if not exponential*, at least very fast convergence for most elliptic problems. These are described in the next few subsections.

1.2.1 Gradual *h* Refinement or *h*/*hp* FEM

This approach requires good discretization and *quasi-uniform meshing.* A *geometric mesh* has been found to be effective with a ratio of 0.15 [11]. The method gives exponential convergence.

The global matrix A can be split into components containing boundary and interior

contributions, that is,

$$A = \begin{bmatrix} A_{II} & A_{IB} \\ (A_{IB})^T & A_{BB} \end{bmatrix},$$

where A_{BB} represents the components of A resulting from boundary-boundary mode interactions, A_{IB} represents the components resulting from coupling between boundary-interior modes and A_{II} represents the components resulting from interior-interior mode interactions.

In the global system the global boundary-boundary, A_{BB}, matrix is sparse and may be reordered to reduce the bandwidth. The elements of A_{BB}, corresponding to the *common boundary values*, are the values along the edges of the elements.

To solve the system $Au = h$ the block L-U factorization of A

$$A = \begin{bmatrix} I & 0 \\ A_{IB}^T A_{II}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{II} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_{II}^{-1} A_{IB} \\ 0 & I \end{bmatrix},$$

where the *Schur complement matrix* S is defined as $S = A_{BB} - A_{IB}^T A_{II}^{-1} A_{IB}$, is used. Consequently, solving $Au = h$ reduces to solving $Su_B = \widetilde{h}_B$, where $\widetilde{h}_B = h_B - A_{IB}^T A_{II}^{-1} h_I$. Once u_B is known we can determine u_I from the equation $u_I = A_{II}^{-1} h_I - A_{II}^{-1} A_{IB} u_B$.

Preconditioning

To solve the algebraic system of equations arising from the discretization of symmetric elliptic BVPs via spectral/*hp* methods we need very efficient and effective preconditioners for *iterative* solvers for large scale computations in parallel environments.

The iterative solution techniques for the linear systems generated from the h- version or p- version of the FEM have been widely studied in the past decades. Among them most successful are the preconditioned conjugate gradient methods. The efficiency of

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these solution techniques relies largely on the condition number of the preconditioned system. For the h- version, Bramble et al. [12] proposed a class of preconditioners based on a representation of the $H_{00}^{1/2}-$ norm on the interfaces of subdomains, which generally results in a condition number of order $O\left(1+\ln\frac{H}{h}\right)^2$, where H and h are the diameters of the subdomains and the elements respectively. On the other hand, Babuska et al. [5] proposed a preconditioner for the p- version of the FEM, based on the orthogonalization procedure. The resulting condition number is of the order $O\left(1+\ln p\right)^2$ with p being the polynomial degree used.

The preconditioning for the h-p version has been considered only recently. In the paper of Guo and Cao [26], a preconditioner for the h-p version of the FEM in two dimensions, with non-uniform mesh and non-uniform distribution of element degrees has been proposed which is also applicable to h-p version associated with the geometric mesh. The condition number of the preconditioned linear system is of the order $\max_i\left(1+\ln\frac{H_i p_i}{h_i}\right)^2$, where H_i is the diameter of the subdomain Ω_i, h_i and p_i are the diameter of elements and the maximum polynomial degree used in Ω_i. This fully covers the h- version when $p_i=1$ and p- version when $h_i=H_i$ as special cases.

1.2.2 Conformal or Auxiliary Mapping

We will consider three different situations for individual equations, namely *Laplace, Poisson* and *Helmholtz* equation.

Laplace Equation

For the Laplace equation $\Delta u=0$, with homogeneous boundary conditions the solution, in the neighborhood of the corner, can be expressed as

$$u\left(r,\theta\right)=\sum_{k=0}^{\infty}a_k\phi_k\left(r,\theta\right),$$

where the coefficients a_k are determined by the boundary conditions and

$$\phi_k(r,\theta) = \begin{cases} r^{k\frac{\pi}{\omega}} \sin\left(k\frac{\pi}{\omega}\theta\right), & \text{if } k\frac{\pi}{\omega} \text{ is not an integer} \\ r^{k\frac{\pi}{\omega}} \left[\ln r \sin\left(k\frac{\pi}{\omega}\theta\right) + \theta \cos\left(k\frac{\pi}{\omega}\theta\right)\right], & \text{if } k\frac{\pi}{\omega} \text{ is an integer} \end{cases}.$$

Here ω is the sectoral angle. Assuming that the logarithmic term doesn't contribute to the solution, the mapping $z = \xi^{\omega/\pi}$, which is conformal at all points except the origin makes the transformed solution analytic in terms of the new variables. Thus

$$u(r,\theta) = \sum_{k=0}^{\infty} a_k r^{k\frac{\pi}{\omega}} \sin\left(k\frac{\pi}{\omega}\theta\right) \mapsto u(\rho,\psi) = \sum_{k=0}^{\infty} a_k \rho^k \sin(k\psi).$$

This method was first implemented in [10] for *hp* finite element discretizations.

Poisson Equation

For the Poisson equation $\Delta u = f(x)$, the situation is more complicated because the solution may not be analytic after the mapping. Typically we decompose the solution into a *homogeneous* part $u^{\mathcal{H}}$, which has the singularity, and a *particular* part $u^{\mathcal{P}}$, which depends on the forcing. Complications arise due to the particular solution since even if it is smooth in the original domain it may be singular after the transformation.
In order to enhance the convergence, we separate the two contributions so that we have an analytic contribution in the z plane and an analytic contribution in the ξ plane.

Helmholtz Equation

For the Helmholtz equation $\Delta u - \lambda u = f(x)$, $\lambda > 0$, the conformal mapping is an effective way of enhancing convergence *although exponential convergence cannot be fully restored.* The auxiliary mapping $z = \xi^{\omega/\pi}$ converts the Helmholtz equation to

$$\Delta u - \lambda \left(\frac{\omega}{\pi}\right)^2 \rho^{2(\omega/\pi-1)} u = \left(\frac{\omega}{\pi}\right)^2 \rho^{2(\omega/\pi-1)} f.$$

In terms of the original variables the solution around the corner is

$$u(r,\theta)=\sum_{k=0}^{\infty}a_k I_{k\frac{\pi}{\omega}}\left(\sqrt{\lambda}r\right)\sin\left(k\frac{\pi}{\omega}\theta\right)$$

for $k\frac{\pi}{\omega}$ not an integer, where $I_m(z)$ is the modified Bessel function of first kind. After application of the mapping the solution has the form

$$u(\rho,\psi)=\sum_{k=0}^{\infty}a_k\rho^k\sin(k\psi)\left(\sum_{j=0}^{\infty}c_j\rho^{2j\frac{\omega}{\pi}}\right)$$

with leading singular term of order $\rho^{1+2\frac{\omega}{\pi}}$. Therefore, the estimated convergence rate is $O\left(P^{-2-4\frac{\omega}{\pi}-\varepsilon}\right)$, which is algebraic but in practice adequately fast.

1.2.3 Singular Basis

An alternative approach to using the auxiliary mapping is to use a set of supplementary basis function which have the leading behavior of the singularity in conjunction with the smooth basis $\Phi_k(x)$. For the Helmholtz equation the leading order singular terms are

$$r^{\frac{\pi}{\omega}},\ r^{2\frac{\pi}{\omega}}\quad(\frac{\omega}{\pi}>1/2),\quad r^{3\frac{\pi}{\omega}}\quad\left(\frac{\omega}{\pi}>1\right),\ldots,$$

which can be included into the expansion basis. However, we can do even better by supplementing the standard basis in the mapped domain. The transformed solution is then

$$u(\rho,\psi)=\sum_{k=1}^{\infty}a_k I_{k\frac{\pi}{\omega}}\left(\sqrt{\lambda}\rho^{\frac{\omega}{\pi}}\right)\sin k\psi=\sum_{k=1}^{\infty}\sum_{l=0}^{\infty}b_{kl}\rho^{k+2l\frac{\omega}{\pi}}\sin k\psi$$

and thus the leading singularities are weaker, i.e.

$$\rho^{1+2\frac{\omega}{\pi}},\ \rho^{2+2\frac{\omega}{\pi}}\quad\left(\frac{\omega}{\pi}>1/2\right),\quad\rho^{1+4\frac{\omega}{\pi}},\ldots.$$

It has been shown in [39] that with one or two terms included in the transformed domain, a very fast convergence is obtained. To achieve the exponential accuracy

we need to include higher order terms but in the limiting case the system becomes *ill-conditioned.*

1.2.4 Steklov Formulation

A more recent method that treats singular solutions of both scalar and vector elliptic problems in the neighborhood of corners has been suggested by Szabo and Babuska. As we have already seen such singular solutions are characterized by the form $u = r^{\beta} f(\theta)$ close to the corner. We consider the given domain and choose the radius of the domains R sufficiently small so that the general form of the solution is still valid and thus

$$\frac{\partial u}{\partial r} = \beta r^{\beta-1} f(\theta) = (\beta/R)\, u \quad \text{on } \Gamma_3$$

where Γ_3 denotes the circular arc of radius R. We consider the Laplacian equation with *Robin* boundary conditions on the segments, which enclose the corner, Γ_1 and Γ_2, that is,

$$au(x) + b\frac{\partial u(x)}{\partial n} = 0, \quad b \neq 0.$$

The variational statement of the Laplace equation with the above boundary conditions is as follows:

Find real numbers β and $u \in H^1(\Omega_R)$ so that

$$a(u,v) + \sum_{i=1}^{2} M_i(u,v) = \beta M_3(u,v) \quad \forall v \in H^1(\Omega_R)$$

where $a(u,v) = (\nabla u, \nabla v)$ is the standard weak Laplacian operator and we have

$$M_i(u,v) = \frac{a}{b}\int_{\Gamma_i} uv ds, \quad i = 1,2,3.$$

We now extend the variational formulation of the problem to a domain that does not include the singular vertex. To this end we consider a cut in the form of circular arc R^*, denoted by Γ_4, that surrounds the corner. The new domain is smaller than before

and doesn't contain any singularities. The *modified Steklov problem* has the form:

Find real numbers β and $u \in H^1(\Omega_{R^*})$ so that

$$a(u,v) + \sum_{i=1}^{2} M_i(u,v) = \beta \left[\sum_{i=2}^{4} M_i(u,v) \right] \quad \forall v \in H^1(\Omega_{R^*}),$$

where $M_4(u,v)$ is defined as the rest of $M_i(u,v)$, $(i \leq 3)$ evaluated on Γ_4. Since the domain Ω_{R^*} does not contain any singularities, standard spectral/*hp* discretization can be used to obtain spectral convergence. This method is set in the *finite element formulation.*

1.3 A Review of the Thesis

Current formulations of spectral methods to solve elliptic problems in non smooth domains allow us to recover only algebraic convergence [14, 28]. One method, which yields relatively fast convergence, makes use of a conformal mapping of the form $z = \xi^\alpha$ to smooth out the singularity that occurs at the corner and is referred to as the method of Auxiliary Mapping. However "*even though the conformal mapping is an effective way of enhancing convergence, exponential convergence cannot be fully recovered*" [28].

A method for obtaining a numerical solution to exponential accuracy for elliptic problems with analytic coefficients posed on curvilinear polygons whose boundary is piecewise analytic with mixed Neumann and Dirichlet boundary conditions, was first proposed by Babuska and Guo [7, 8] within the framework of the FEM. They were able to resolve the singularities which arise at the corners by using a geometrical mesh.

We also use a geometrical mesh to solve the same class of problems to exponential accuracy using *h-p* Spectral element methods but with an important difference. With this mesh we seek a solution which minimizes a weighted L^2 norm of the residuals in the partial differential equation and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity by adding a term which measures the jump in the function and its derivatives at *inter-element* boundaries, in an appropriate frac-

tional Sobolev norm, to the functional being minimized. Since the second derivatives of the actual solution are not square integrable in a neighborhood of the corners we have to multiply the residuals in the partial differential equation by an appropriate power of r_k, where r_k measures the distance between the point P and the vertex A_k in a sectoral neighborhood of each of these vertices. In each of these sectoral neighborhoods we use a local coordinate system (τ_k, θ_k) where $\tau_k = \ln r_k$ and (r_k, θ_k) are polar coordinates with origin at A_k. We then derive *differentiability estimates* with respect to these new variables and a *stability estimate* for the functional we minimize. All these quantities are computed in different coordinate systems consisting of local coordinate systems in a sectoral neighborhood of each of the vertices and a global one outside these neighborhoods. Thus we use the *auxiliary map* $z = \log \xi$ to solve the problem with exponential accuracy.

We then use the stability estimate to obtain *parallel preconditioners* and *error estimates* for the solution of the minimization problem which are nearly optimal as the condition number of the preconditioned system is *polylogarithmic* in N, i.e. $O\left(1+\ln N\right)^2$, where N is the number of processors and the number of degrees of freedom in each variable on each element. Moreover if the data is analytic then the error is *exponentially small* in N.

The method we propose is essentially a *least-squares* method. To compute the residuals in the *normal equations*, however, we do not need to compute any *mass and stiffness matrices* nor do we need to filter the coefficients of the differential and boundary operators or the data. We solve the resulting normal equations by the preconditioned conjugate gradient method.

To solve elliptic boundary value problems with mixed Neumann and Dirichlet boundary conditions the spectral element functions we use are *not fully non-conforming*; instead the spectral element functions are restricted so that their values at the common vertices of the elements on which they are defined are the same. We divide the vector composed of the values of the spectral element functions at the Legendre-Gauss-

Lobatto points on each element into two sub vectors - one consisting of the values of the spectral element functions at the vertices of all the elements and the other consisting of the remaining values. The dimension of the first sub vector, which can be thought of as a vector of common boundary values is very small compared to FEM.

The normal equations thus obtained are solved by the preconditioned conjugate gradient method. We first solve a much smaller system of equations corresponding to the *Schur complement* of the sub vector of common boundary values. The Schur complement matrix can be computed accurately since it's dimension is small. Moreover, the error in the numerical solution is exponentially small in N, which is proportional to the number of elements and the degree of the spectral element representations in each variable on each element.

We finally show that the h-p Spectral element method applies to elliptic problems on *curvilinear polygons* with mixed Neumann and Dirichlet boundary conditions provided the Babuska-Brezzi *inf-sup* conditions are satisfied.

We now briefly describe the contents of this dissertation. In Chapter 1 we provide a review of existing work. In Chapters 2 and 3 we restrict ourselves to examining the Poisson equation with Dirichlet boundary conditions on a polygon. In Chapter 2 we obtain differentiability estimates in modified polar coordinates and prove the stability theorem 2.3 on which our method is based. Since the statement of this theorem may appear complicated we try and provide motivation for it by stating the stability theorem 2.1 for a simpler case. In Chapter 3 we provide computational techniques and error estimates for Dirichlet problems.

In Chapter 4 we examine how to solve the Poisson equation with mixed Dirichlet and Neumann boundary conditions. In both Chapters 3 and 4 we also provide numerical results.

Finally in Chapter 5 we generalize all our results to elliptic problems with analytic coefficients, posed on curvilinear polygons with piecewise analytic boundaries, which satisfy the Babuska-Brezzi inf-sup conditions.

Chapter 2

Differentiability and Stability Estimates for Dirichlet Problems

2.1 Introduction

To overcome the singularities that arise in a neighborhood of the corners we use a *geometrical mesh*, as has been proposed by Babuska and Guo. The geometrical mesh becomes geometrically fine in a neighborhood of each of the corners. In a neighborhood of the corner A_k we switch to new variables (τ_k, θ_k) where $\tau_k = \ln r_k$ and (r_k, θ_k) are polar coordinates with origin at A_k. In doing so the geometrical mesh is reduced to a *quasi-uniform mesh* in a sectoral neighborhood of the corners and so *Sobolev's embedding theorems* and the *trace theorems* for Sobolev spaces apply for functions defined on mesh elements in these new variables with a ***Uniform Constant***. These new variables, which we shall refer to as *modified polar coordinates*, were first used by Kondratiev in his seminal paper [30]. Away from these sectoral neighborhoods of the corners we retain (x, y) variables for our coordinate system. Thus we also use the *auxiliary map* $z = \log \xi$ to remove the singularity at the origin and this enables us to obtain the solution with spectral accuracy.

By subtracting an analytic function from the solution if necessary, we may assume

that the Dirichlet data vanishes at the corners. We seek an approximate solution which *vanishes* at the corner-most elements and is a sum of *tensor product* of polynomials of degree N in τ_k and θ_k in the remaining elements of the sectoral neighborhood of the corners. The remaining quadrilateral elements are mapped to the unit square S and the approximate solution is represented as a sum of tensor products of polynomials of degree N in ξ and η, the transformed variables. If Neumann boundary conditions are imposed on both the sides which meet at the corner, the approximate solution at corner most elements is represented by a constant, instead of zero.

We now seek a solution as in [17] which minimizes the sum of the squares of a weighted L^2 norm of the residuals in the partial differential equation and the sum of the squares of the residuals in the Dirichlet boundary conditions in an appropriate Sobolev norm and enforce continuity by adding a term which measure the sum of the squares of the jump in the function in the L^2 norm and the squares of the jump in its derivatives across *inter-element boundaries* in an appropriate *fractional Sobolev norm* to the functional being minimized. Since the residuals in the partial differential equation blow up in a neighborhood of the corners, we have to multiply these residuals by an appropriate power of r_k, where r_k measures the distance between the point P and A_k. All these computations are done using modified polar coordinates in a sectoral neighborhood of the corners and a global coordinate system elsewhere.

For the Dirichlet problem we use Spectral element functions which are *non-conforming*. To solve the minimization problem we have defined, we need to solve the *normal equations* for the *least-squares* problem corresponding to collocating the partial differential equation and boundary conditions at an *over-determined* set of collocation points and enforcing continuity of the function and its derivatives at the collocation points at inter-element boundaries, suitably weighted. However we do not need to compute and store any matrices, like the *mass and stiffness matrices*, to compute the residual in the normal equations [18].

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We can precondition the normal equations by using a preconditioner which is of

block diagonal form and which allows the solutions for different elements to decouple completely. Moreover this is nearly optimal as the condition number of the preconditioned system is polylogarithmic in N, which is proportional to the number of processors and the number of degrees of freedom in each variable on each element. Finally we show that the error we commit is *exponentially small* in N.

2.2 Function Spaces and A'Priori Estimates

Let Ω be a polygon with vertices $A_1, A_2, \ldots, A_p$ and corresponding sides $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ where Γ_i joins the points A_{i-1} and A_i. In addition let the angle subtended at A_j be ω_j.

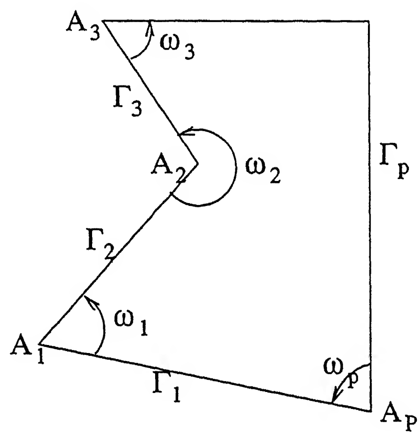

Figure 2.1: Polygonal domain

In this chapter we shall examine the solution of the problem

$$\Delta u = f \qquad \text{for } (x,y) \in \Omega, \tag{2.1a}$$

with Dirichlet boundary conditions

$$\begin{aligned} u &= g_j \qquad \text{for } (x,y) \in \Gamma_j, \text{ or} \\ u &= g^{[0]} \qquad \text{for } (x,y) \in \Gamma^{[0]} = \partial\Omega. \end{aligned} \tag{2.1b}$$

Let Z denote the point $Z = (x,y)$.

We now need to review a set of *a'priori* estimates proved in [7]. Let $H^m(\Omega)$ denote the

completion of the space of infinitely differentiable functions with respect to the norm

$$\|v\|_{m,\Omega}^2 = \sum_{|\alpha|\leq m} \int\int |D^\alpha v|^2 \, dxdy.$$

Let ρ_i denote the Euclidean distance between A_i and Z, i.e. $\rho_i = |Z - A_i|$; we then define $r_i = \min(1, \rho_i)$. We shall let β denote the multi index $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$. Further, we define

$$\Phi_\beta (Z) = \Pi_{i=1}^p r_i^{\beta_i} (Z).$$

By $H_\beta^{m,l} (\Omega)$ we denote the completion of infinitely differentiable functions with respect to the norm

$$\|v\|_{H_\beta^{m,l}(\Omega)}^2 = \|v\|_{H^{l-1}(\Omega)}^2 + \sum_{k=l,|\alpha|=k}^{m} \|D^\alpha u \Phi_{\beta+k-l}\|_{L^2(\Omega)}^2, l \geq 1.$$

Let $H_\beta^{m-1/2,l-1/2} (\Gamma_j)$ be the space of functions ϕ_j such that there exists $f \in H_\beta^{m,l} (\Omega)$ so that $f|_{\Gamma_j} = \phi_j$ and define

$$\|\phi_j\|_{H_\beta^{m-1/2,l-1/2}(\Gamma_j)} = \inf_{f\in H_\beta^{m,l}(\Omega)} \|f\|_{H_\beta^{m,l}(\Omega)}.$$

Let

$$\psi_\beta^l (\Omega) = \left\{ u (Z) \middle| u \in H_\beta^{m,l} (\Omega), \ m \geq l \right\}$$

and

$$\mathfrak{B}_\beta^l (\Omega) = \left\{ \begin{array}{c} u (Z) \Big| u \in \psi_\beta^l (\Omega), \| |D^\alpha u| \, \Phi_{\beta+k-l} \|_{L^2(\Omega)} \leq C d^{k-l} (k-l)!, \\ \text{for } |\alpha| = k = l, l+1, \ldots, d > 1, \text{ C independent of k} \end{array} \right\}.$$

Let $Q \subseteq \mathbb{R}^2$ be an open set with a piecewise analytic boundary ∂Q and let γ be part or whole of the boundary ∂Q. Finally let $\mathfrak{B}_\beta^{l-1/2} (\gamma), 0 \leq l \leq 2$, be the space of all

functions φ for which there exists $f \in \mathfrak{B}^l_\beta(Q)$ such that $f = \varphi$ on γ.

We now cite the important regularity theorem 2.1 of [7]. Let $f \in \mathfrak{B}^0_\beta$, $g^{[0]} \in \mathfrak{B}^{3/2}_\beta\left(\Gamma^{[0]}\right)$, $\beta = (\beta_1, \beta_2, \ldots, \beta_p)$, $0 < \beta_i < 1$, $\beta_i > 1 - \pi/\omega_i$. Then the problem (2.1a - 2.1b) has a unique solution in $H^1(\Omega)$ and $u \in \mathfrak{B}^2_\beta(\Omega)$.

Now in Section 4 of [8] it has been shown that when $g^{[0]}$ is analytic on every closed arc $\overline{\Gamma}_i$ and $g^{[0]}$ is continuous on $\Gamma^{[0]}$ then $g^{[0]} \in \mathfrak{B}^{3/2}_\beta\left(\Gamma^{[0]}\right)$. Further if f is analytic then it belongs to $\mathfrak{B}^0_\beta$.

Next as in [8] we introduce the space $\mathfrak{C}^2_\beta$:

$$\mathfrak{C}^2_\beta = \left\{ \begin{array}{c} u \in H^{2,2}_\beta(\Omega) \left| |D^\alpha u(Z)| \leq C d^k k! \left(\Phi_{k+\beta-1}(Z)\right)^{-1}, \right. \\ |\alpha| = k = 1, 2, \ldots, \ C \geq 1,\ d \geq 1 \text{ independent of k} \end{array} \right\}.$$

The relationship between $\mathfrak{C}^2_\beta$ and $\mathfrak{B}^2_\beta$ is given by Theorem 2.2 of [8] which we state as follows:

$$\mathfrak{B}^2_\beta(\Omega) \subseteq \mathfrak{C}^2_\beta.$$

Finally we need one last result from [8], viz. Lemma 2.1 which is stated below: Let $u \in H^{2,2}_\beta(\Omega)$. Then u is continuous on $\overline{\Omega}$ and

$$\|u\|_{C(\overline{\Omega})} \leq C \|u\|_{H^{2,2}_\beta(\Omega)}.$$

Since we are assuming that the data, $g_1, \ldots, g_p$ are analytic and compatible at the vertices the values of u at the vertices $A_1, A_2, \ldots, A_p$ are well defined. Thus if we subtract from u an analytic function which assumes these values at the vertices then the difference would satisfy (2.1a - 2.1b) with a modified set of analytic data and the Dirichlet boundary data would assume the value zero at $A_1, A_2, \ldots, A_p$. Hence without loss of generality we may assume $g_{j-1}(A_j) = g_j(A_j) = 0$ for $j > 1$ and $g_p(A_1) = 0$.

We now define one last norm which will be needed in the sequel. Let

$$\|u(\tau_j,\theta_j)\|^2_{m,(-\infty,\ln\mu)\times(\psi_l^j,\psi_u^j)} = \sum_{|\alpha|\le m}\int_{\psi_l^j}^{\psi_u^j}\int_{-\infty}^{\ln\mu}\left|D_{\tau_j}^{\alpha_1}D_{\theta_j}^{\alpha_2}u\right|^2 d\tau_j d\theta_j.$$

We need to obtain an asymptotic estimate on $\|u(\tau_j,\theta_j)\|^2_{m,(-\infty,\ln\mu)\times(\psi_l^j,\psi_u^j)}$ as $\mu\to 0$. It is easy to estimate the terms in the right hand side of the above definition when $|\alpha|\ge 2$. For

$$\begin{aligned}
(2.2)\qquad & \sum_{2\le|\alpha|\le m}\int_{\psi_l^j}^{\psi_u^j}\int_{-\infty}^{\ln\mu}\left(u_{\tau_j^{\alpha_1}\theta_j^{\alpha_2}}\right)^2 d\tau_j d\theta_j \\
\le\ & \mu^{2(1-\beta_j)}\sum_{2\le|\alpha|\le m}\int_{\psi_l^j}^{\psi_u^j}\int_{-\infty}^{\ln\mu}\left(u_{\tau_j^{\alpha_1}\theta_j^{\alpha_2}}\right)^2 e^{-2(1-\beta_j)\tau_j}d\tau_j d\theta_j \\
\le\ & \mu^{2(1-\beta_j)}\sum_{2\le|\alpha|\le m}\int_{\psi_l^j}^{\psi_u^j}\int_{0}^{\mu}(r_j)^{2\alpha_1}\left(u_{r_j^{\alpha_1}\theta_j^{\alpha_2}}\right)^2\left(r_j^{-2+\beta_j}\right)^2 r_j dr_j d\theta_j \\
\le\ & \mu^{2(1-\beta_j)}\left(Cd^{m-2}(m-2)!\right)^2.
\end{aligned}$$

Here we have used the fact that $u\in\mathfrak{B}_\beta^2(\Omega)$ and Theorem 1.1 of [7] to obtain the above result. Next we bound the terms when $|\alpha|=1$. Since $u\in\mathfrak{C}_\beta^2(\Omega)$ we have

$$(2.3)\qquad |D^\alpha u(Z)|\le Cd\left(\Phi_\beta(Z)\right)^{-1} \text{ when } |\alpha|=1.$$

Moreover we have

$$\int_{\psi_l^j}^{\psi_u^j}\int_{-\infty}^{\ln\mu}\left(u_{\tau_j}^2+u_{\theta_j}^2\right)d\tau_j d\theta_j = \int_{S_j^\mu}\int\left(u_x^2+u_y^2\right)dxdy.$$

Here

$$S_j^\mu=\left\{(x,y)\,\middle|\,0<r_j<\mu,\ \psi_l^j<\theta_j<\psi_u^j\right\}.$$

Hence by the above relation

$$(2.4)\qquad \int_{S_j^\mu}\int\left(u_x^2+u_y^2\right)dxdy\le 2C^2d^2\int_{\psi_l^j}^{\psi_u^j}\int_0^\mu r_j^{-2\beta_j}r_j dr_j d\theta_j\le (Kd)^2\mu^{2(1-\beta_j)}.$$

Finally we have to estimate

$$\int_{\psi_l^j}^{\psi_u^j} \int_{-\infty}^{\ln \mu} |u(\tau_j, \theta_j)|^2 \, d\tau_j d\theta_j.$$

Since u vanishes at A_j

$$u(\tau_j, \theta_j) = \int_{-\infty}^{\tau_j} u_\eta(\eta, \theta_j) \, d\eta.$$

Hence

$$|u(\tau_j, \theta_j)| \leq \left| \int_0^{r_j} u_\rho(\rho, \theta_j) \, d\rho \right|.$$

Here $\rho = e^\eta$ and $r_j = e^{\tau_j}$. Since $u \in \mathfrak{C}^2_\beta(\Omega)$ we obtain

$$|u(\tau_j, \theta_j)| \leq C d r_j^{-\beta_j + 1} = C d e^{(1-\beta_j)\tau_j}.$$

And integrating the above with respect to τ_j and θ_j gives

$$\int_{\psi_l^j}^{\psi_u^j} \int_{-\infty}^{\ln \mu} |u(\tau_j, \theta_j)|^2 \, d\tau_j d\theta_j \leq (Kd)^2 \mu^{2(1-\beta_j)}. \tag{2.5}$$

Combining (2.2), (2.4) and (2.5) we get the required estimate

$$\sum_{|\alpha| \leq m} \int_{\psi_l^j}^{\psi_u^j} \int_{-\infty}^{\ln \mu} \left| u_{\tau_j^{\alpha_1} \theta_j^{\alpha_2}} \right|^2 d\tau_j d\theta_j \;\; \leq \;\; \mu^{2(1-\beta_j)} \left(C d^{m-2} (m-2)! \right)^2. \tag{2.6}$$

2.3 Stability Estimates

We first need to divide Ω into subdomains. Thus we divide Ω into p subdomains $S^1, S^2, \ldots, S^p$, where S^i denotes a domain which contains the vertex A_i and no other, and on each S^i we define a geometric mesh as has been done in [8].

Let $\mathfrak{S}^k = \{\Omega^k$ ...

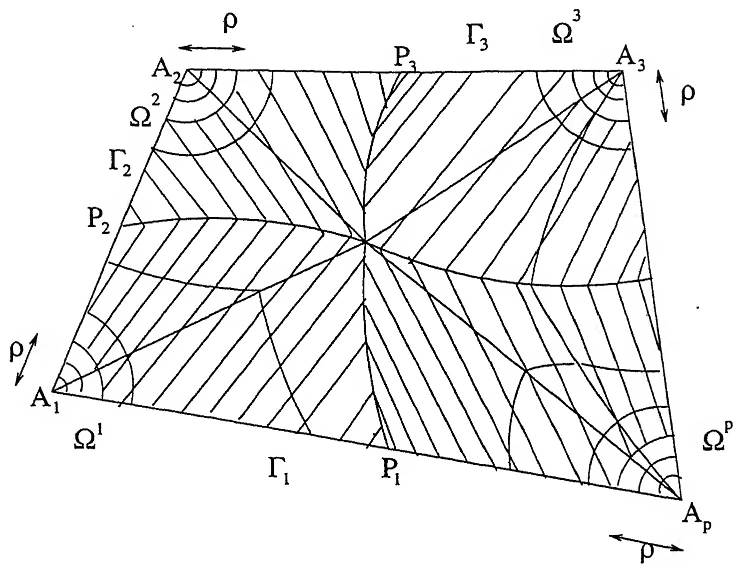

Figure 2.2: Mesh over the whole domain

$\bigcup_{k=1}^{p} \mathfrak{S}^k$. Then $\mathfrak{S}$ satisfies the following conditions:

1. $\Omega_{i,j}^k$ are curvilinear quadrilaterals or triangles and the intersection of any two $\Omega_{i,j}^k$ is one common vertex or one entire side or is empty.

2. Let $h_{i,j}^k$ and $\underline{h}_{i,j}^k$ be the maximal and minimal length of the sides of $\Omega_{i,j}^k$. We shall assume there is a constant independent of i, j, k and of the partition such that

$$\frac{h_{i,j}^k}{\underline{h}_{i,j}^k} \leq \lambda. \tag{2.7}$$

3. Let $M = \left\{M_{i,j}^k,\ 1 \leq i \leq I_{k,j},\ 1 \leq j \leq J_k,\ 1 \leq k \leq p\right\}$ in which $M_{i,j}^k$ is a one-to-one mapping of the closed standard master square $S = [0,1] \times [0,1]$ {respectively standard master triangle $T = \{(\xi, \eta)\,|0 \leq \eta \leq 1 - \xi,\ 0 \leq \xi \leq 1\}$ onto $\overline{\Omega}_{i,j}^k$. Let $P_{i,j,l}^k$ and $\gamma_{i,j,l}^k$ denote the vertices and sides of $\Omega_{i,j}^k$, then $\left(M_{i,j}^k\right)^{-1}\left(P_{i,j,l}^k\right)$ and $\left(M_{i,j}^k\right)^{-1}\left(\gamma_{i,j,l}^k\right)$ denote the vertices and sides of S (respectively T), $1 \leq l \leq 4$ (respectively $1 \leq l \leq 3$). Moreover if $M_{i,j}^k$ and $M_{m,n}^l$ map the closed standard square S onto elements $\overline{\Omega}_{i,j}^k$ and $\overline{\Omega}_{m,n}^l$ with common side $\gamma = \overline{P_1 P_2}$, then for any $P \in \gamma$, $\operatorname{dist}\left(\left(M_{i,j}^k\right)^{-1}(P), \left(M_{i,j}^k\right)^{-1}(P_t)\right) = \operatorname{dist}\left(\left(M_{m,n}^l\right)^{-1}(P), \left(M_{m,n}^l\right)^{-1}(P_t)\right)$,

$1 \le t \le 2$. We will assume $M_{i,j}^k$ can be written in the form

$$x = X_{i,j}^k(\xi,\eta), \qquad (\xi,\eta) \in S \text{ (respectively T)} \tag{2.8}$$
$$y = Y_{i,j}^k(\xi,\eta),$$

with $X_{i,j}^k$ and $Y_{i,j}^k$ being analytic functions on S (respectively T). Further we assume that for $|\alpha| \le 2$

$$|D^\alpha x|, |D^\alpha y| \le C h_{i,j}^k, \text{ and} \tag{2.9a}$$
$$C_1 \left(h_{i,j}^k\right)^2 \le J_{i,j}^k \le C_2 \left(h_{i,j}^k\right)^2 \tag{2.9b}$$

for all i, j and k with constants C, C_1, C_2 independent of i, j and k and $J_{i,j}^k$ being the Jacobian of the mapping $M_{i,j}^k$.

Let $\mu = (\mu_1, \dots, \mu_p)$ with $0 < \mu_i < 1$. Then $\mathfrak{S}_\mu$ is called a geometrical mesh with ratios $\mu = (\mu_1, \dots, \mu_p)$ when in addition the following condition is fulfilled.

4. Let $\Omega_{i,j}^k \in \mathfrak{S}$ and $d_{i,j}^k$ denote the distance between $\Omega_{i,j}^k$ and A_k. Then $d_{i,j}^k$ and $h_{i,j}^k$ satisfy

$$C_1 (\mu_k)^{N-j} \le d_{i,j}^k \le C_2 (\mu_k)^{N-j}, \ 1 < j \le N, \ 1 \le i \le I_{k,j}, \tag{2.10a}$$
$$C_3 \rho \le d_{i,j}^k \le C_4 \rho, \ N < j \le J_k, \ 1 \le i \le I_{k,j}, \ 1 \le k \le p, \tag{2.10b}$$
$$d_{i,1}^k = 0, \ 1 \le i \le I_{k,1}, \ 1 \le k \le p, \tag{2.10c}$$
$$K_1 d_{i,j}^k \le \underline{h}_{i,j}^k \le h_{i,j}^k \le K_2 d_{i,j}^k \tag{2.10d}$$

for $1 < j \le J_k$, $1 \le i \le I_{k,j}$, $1 \le k \le p$, where C_l for $1 \le l \le 4$ and K_l for $1 \le l \le 2$ are constants independent of i, j and k. Moreover $J_k = N + O(1)$.

We now put some restrictions on $\mathfrak{S}$. Let (r_k, θ_k) denote polar coordinates with center at A_k. Let $\tau_k = \ln r_k$. We choose ρ so that the sector S_ρ^k with sides Γ_k and Γ_{k+1}, center

at A_k and radius ρ satisfies

$$S_\rho^k \subseteq \bigcup_{\Omega_{i,j}^k \in \mathfrak{S}^k} \overline{\Omega}_{i,j}^k.$$

S_ρ^k may be represented as

$$S_\rho^k = \left\{(x,y)\,\middle|\,0 < r_k < \rho,\ \psi_l^k < \theta_k < \psi_u^k\right\}. \tag{2.11a}$$

Let $\left\{\psi_i^k\right\}_{i=1,\dots,I_{k+1}}$ be an increasing sequence of points such that $\psi_1^k = \psi_l^k$ and $\psi_{I_{k+1}}^k = \psi_u^k$. Let $\Delta\psi_i^k = \psi_{i+1}^k - \psi_i^k$. We choose these points so that

$$\max_k \left(\max_i \Delta\psi_i^k\right) \le \lambda \min_k \left(\min_i \Delta\psi_i^k\right) \tag{2.11b}$$

for some constant λ. Let

$$\sigma_1^k = 0, \text{ and} \tag{2.11c}$$

$$\sigma_j^k = \rho\left(\mu_k\right)^{N+1-j} \text{ for } 2 \le j \le N+1. \tag{2.11d}$$

Finally we define

$$\eta_j^k = \ln \sigma_j^k \text{ for } 1 \le j \le N+1. \tag{2.12}$$

Let

$$\Omega_{i,j}^k = \left\{(x,y)\,\middle|\,\sigma_j^k < r_k < \sigma_{j+1}^k,\ \psi_i^k < \theta_k < \psi_{i+1}^k\right\}, \tag{2.13}$$

for $1 \le i \le I_k$, $1 \le j \le N$.

We assume there exists a number ν such that

$$\Omega_{i,N+1}^k = \left\{(x,y)\,\middle|\,\rho < r_k < \nu,\ \psi_i^k < \theta_k < \psi_{i+1}^k\right\}, \tag{2.14}$$

for $1 \leq i \leq I_k$.

In other words $I_{k,j}$ is independent of j for $j \leq N+1$. We shall let O^k denote

$$O^k = \left\{ \Omega^k_{i,j} \,|1 \leq i \leq I_k,\ 1 \leq j \leq N \right\}, \tag{2.15}$$

for $1 \leq k \leq p$.

Let

$$O^{p+1} = \left\{ \Omega^k_{i,j} \,|1 \leq i \leq I_{k,j},\ N+1 \leq j \leq J_k,\ 1 \leq k \leq p \right\}. \tag{2.16}$$

We shall relabel the elements of O^{p+1} and write

$$O^{p+1} = \left\{ \Omega^{p+1}_l \,|1 \leq l \leq L \right\}, \tag{2.17}$$

where L denotes the *cardinality* of O^{p+1}. We shall let Ω^k denote the sector with vertex at A_k given by

$$\Omega^k = \left\{ (x,y) \,|0 < r_k < \rho,\ \psi^k_l < \theta_k < \psi^k_u \right\}, \text{ and} \tag{2.18a}$$

$$\Omega^{p+1} = \Omega \Big\backslash \left\{ \bigcup_{k=1}^{p} \overline{\Omega}^k \right\}. \tag{2.18b}$$

Notice that all the sets Ω^k are open sets.

Henceforth to keep our notation simple we will assume that $\Omega^k_{i,j}$ are quadrilaterals for $1 \leq k \leq p$, $N+1 \leq j \leq J_k$, $1 \leq i \leq I_k$. Moreover we assume $I_{k,j} \leq I$ for all k and j. Here I is a small integer, and this fact plays a fundamental role in allowing us to use non-conforming spectral elements to solve the Dirichlet problem. Further let $\gamma^k_{i,j,l}$ denote the sides of the quadrilateral $\Omega^k_{i,j}$, $1 \leq l \leq 4$. Then we assume

$$\gamma^k_{i,j,l} : \begin{cases} x = h^k_{i,j}\phi^k_{i,j,l}(\xi), \\ y = h^k_{i,j}\psi^k_{i,j,l}(\xi), \end{cases} \quad 0 \leq \xi \leq 1,\ l = 1,3 \tag{2.19}$$

$$\gamma^k_{i,j,l} : \begin{cases} x = h^k_{i,j}\phi^k_{i,j,l}(\eta), \\ y = h^k_{i,j}\psi^k_{i,j,l}(\eta), \end{cases} \quad 0 \le \eta \le 1,\ l = 2,4 \tag{2.20}$$

where $\phi^k_{i,j,l}$, $\psi^k_{i,j,l}$ are analytic functions for all i, j, k and l. We wish to obtain a *stability estimate* for the function u, given by a non-conforming finite dimensional representation $u^k_{i,j}$ on each domain $\Omega^k_{i,j}$, for the entire polygonal domain Ω. Now, as stated in the introduction we partition the open set Ω into p open sets $S^1, S^2, \ldots, S^p$ such that each S^i contains only the singularity at the vertex A_i. Let S^k be one of these open sets. Then $S^k = \Omega^k \bigcup B^k_\rho \bigcup T^k$ where Ω^k is the open sector with center at A^k and radius ρ, B^k_ρ is the circular arc which bounds Ω^k, and T^k is the open set defined as $T^k = S^k \backslash (\Omega^k \bigcup B^k_\rho)$.

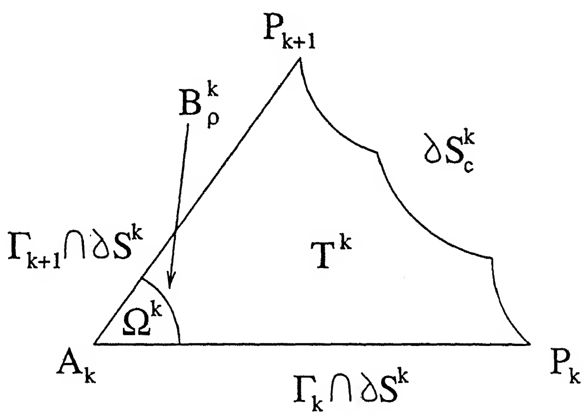

Figure 2.3: Open set S^k

The domain S^k is as shown in Fig. 2.3. Two of its sides are the straight lines $\Gamma_{k+1} \bigcap \partial S^k$ and $\Gamma_k \bigcap \partial S^k$. The remaining side ∂S^k_c consists of piecewise analytic arcs. The subscript c in ∂S^k_c denotes curvilinear. $\overline{S}^k$ is partitioned by a set of arcs $\{\gamma_l\}_l$ into subdomains.

Now

$$\Omega^k_{i,j} = \left\{(x,y) \left| \sigma^k_j < r_k < \sigma^k_{j+1},\ \psi^k_i < \theta_k < \psi^k_{i+1} \right.\right\}$$

for $1 \le i \le I_k,\ 1 \le j \le N$. Let $\tau_k = \ln r_k$. Define

$$\widetilde{\Omega}^k_{i,j} = \left\{ (\tau_k, \theta_k) \,\middle|\, \eta^k_j < \tau_k < \eta^k_{j+1},\ \psi^k_i < \theta_k < \psi^k_{i+1} \right\} \tag{2.21}$$

for $1 \le i \le I_k,\ 1 \le j \le N$. Let ω be a smooth function. Then

$$\omega_{xx} + \omega_{yy} = \frac{1}{r_k^2}\left(r_k \frac{\partial}{\partial r_k} r_k \frac{\partial \omega}{\partial r_k} + \omega_{\theta_k\theta_k} \right) = e^{-2\tau_k}\left(\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k} \right).$$

Hence

$$\int_{\Omega^k_{i,j}}\!\!\int r_k^2 \left(\omega_{xx} + \omega_{yy}\right)^2 dxdy = \int_{\psi^k_i}^{\psi^k_{i+1}} \int_{\eta^k_j}^{\eta^k_{j+1}} \left(\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k}\right)^2 d\tau_k d\theta_k.$$

Now using integration by parts repeatedly we can show that

$$\begin{aligned}
&\int_{\psi^k_i}^{\psi^k_{i+1}} \int_{\eta^k_j}^{\eta^k_{j+1}} \left(\left(\omega_{\tau_k\tau_k}\right)^2 + 2\left(\omega_{\tau_k\theta_k}\right)^2 + \left(\omega_{\theta_k\theta_k}\right)^2 \right) d\tau_k d\theta_k \\
= &\int_{\psi^k_i}^{\psi^k_{i+1}} \int_{\eta^k_j}^{\eta^k_{j+1}} \left(\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k}\right)^2 d\tau_k d\theta_k + 2\int_{\psi^k_i}^{\psi^k_{i+1}} \omega_{\tau_k\theta_k}\omega_{\theta_k}\left(\eta^k_{j+1}, \theta_k\right) d\theta_k \\
- \ &2\int_{\psi^k_i}^{\psi^k_{i+1}} \omega_{\tau_k\theta_k}\omega_{\theta_k}\left(\eta^k_j, \theta_k\right) d\theta_k - 2\int_{\eta^k_j}^{\eta^k_{j+1}} \omega_{\tau_k\tau_k}\omega_{\theta_k}\left(\tau_k, \psi^k_{i+1}\right) d\tau_k \\
+ \ &2\int_{\eta^k_j}^{\eta^k_{j+1}} \omega_{\tau_k\tau_k}\omega_{\theta_k}\left(\tau_k, \psi^k_i\right) d\tau_k.
\end{aligned} \tag{2.22}$$

Moreover

$$\begin{aligned}
&\int_{\Omega^k_{i,j}}\!\!\int -\omega\left(\omega_{xx} + \omega_{yy}\right) dxdy \\
= &-\int_{\psi^k_i}^{\psi^k_{i+1}} \int_{\eta^k_j}^{\eta^k_{j+1}} \omega\left(\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k}\right) d\tau_k d\theta_k \\
= &\int_{\psi^k_i}^{\psi^k_{i+1}} \int_{\eta^k_j}^{\eta^k_{j+1}} \left(\left(\omega_{\tau_k}\right)^2 + \left(\omega_{\theta_k}\right)^2 \right) d\tau_k d\theta_k - \int_{\psi^k_i}^{\psi^k_{i+1}} \omega\omega_{\tau_k}\left(\eta^k_{j+1}, \theta_k\right) d\theta_k \\
+ &\int_{\psi^k_i}^{\psi^k_{i+1}} \omega\omega_{\tau_k}\left(\eta^k_j, \theta_k\right) d\theta_k - \int_{\eta^k_j}^{\eta^k_{j+1}} \omega\omega_{\theta_k}\left(\tau_k, \psi^k_{i+1}\right) d\tau_k + \int_{\eta^k_j}^{\eta^k_{j+1}} \omega\omega_{\theta_k}\left(\tau_k, \psi^k_i\right) d\tau_k.
\end{aligned}$$

And this gives us the following inequality

$$
\begin{aligned}
(2.23)\qquad & \int_{\psi_i^k}^{\psi_{i+1}^k} \int_{\eta_j^k}^{\eta_{j+1}^k} \left((\omega_{\tau_k})^2 + (\omega_{\theta_k})^2 \right) d\tau_k d\theta_k \\
\leq \; & \frac{K}{2} \int_{\psi_i^k}^{\psi_{i+1}^k} \int_{\eta_j^k}^{\eta_{j+1}^k} \omega^2 d\tau_k d\theta_k + \frac{1}{2K} \int_{\psi_i^k}^{\psi_{i+1}^k} \int_{\eta_j^k}^{\eta_{j+1}^k} \left(\omega_{\tau_k\tau_k} + \omega_{\theta_k\theta_k} \right)^2 d\tau_k d\theta_k \\
+ \; & \int_{\psi_i^k}^{\psi_{i+1}^k} \omega\omega_{\tau_k} \left(\eta_{j+1}^k, \theta_k \right) d\theta_k - \int_{\psi_i^k}^{\psi_{i+1}^k} \omega\omega_{\tau_k} \left(\eta_j^k, \theta_k \right) d\theta_k \\
+ \; & \int_{\eta_j^k}^{\eta_{j+1}^k} \omega\omega_{\theta_k} \left(\tau_k, \psi_{i+1}^k \right) d\tau_k - \int_{\eta_j^k}^{\eta_{j+1}^k} \omega\omega_{\theta_k} \left(\tau_k, \psi_i^k \right) d\tau_k .
\end{aligned}
$$

Let $u_{i,j}^k(\tau_k, \theta_k)$ be a set of non-conforming elements, defined on $\widetilde{\Omega}_{i,j}^k$ the image of $\Omega_{i,j}^k$ in (τ_k, θ_k) coordinates, given by

$$
u_{i,j}^k(\tau_k, \theta_k) = \sum_{n=0}^{N} \sum_{m=0}^{N} a_{m,n} \tau_k^m \theta_k^n
$$

for $j > 1$. We shall choose $u_{i,1}^k \equiv 0$ for all k and i. Let

$$
\begin{aligned}
\left[u_{i,j}^k\right] \left(\eta_{j+1}^k, \theta_k \right) &= \left(u_{i,j+1}^k - u_{i,j}^k \right) \left(\eta_{j+1}^k, \theta_k \right), \\
\left[u_{i,j}^k\right] \left(\tau_k, \psi_{i+1}^k \right) &= \left(u_{i+1,j}^k - u_{i,j}^k \right) \left(\tau_k, \psi_{i+1}^k \right).
\end{aligned}
$$

denote the jump in u across inter element boundaries.

We now state and prove a stability theorem which will help to motivate the stability theorem 2.3 on which our numerical scheme is based.

Theorem 2.1 *For the sectoral domain $\widetilde{\Omega}^k$, the following stability estimate holds.*

$$
\begin{aligned}
(2.24)\qquad & \sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| \left(u_{i,j}^k \right) (\tau_k, \theta_k) \right\|_{2,\widetilde{\Omega}_{i,j}^k}^2 \\
\leq \; & C (\ln V)^2 \left\{ \sum_{j=2}^{N} \sum_{i=1}^{I_k} \int_{\psi_i^k}^{\psi_{i+1}^k} \int_{\eta_j^k}^{\eta_{j+1}^k} \left(\left(u_{i,j}^k \right)_{\tau_k\tau_k} + \left(u_{i,j}^k \right)_{\theta_k\theta_k} \right)^2 d\tau_k d\theta_k \right. \\
& \left(\left\| \left[\left(u_{i,j}^k \right)_{\tau_k} \right] \left(\eta_{j+1}^k, \theta_k \right) \right\|_{\frac{1}{2}, (\psi_i^k, \psi_{i+1}^k)}^2 + \left\| \left[\left(u_{i.j}^k \right) \right] \left(\eta_{j+1}^k, \theta_k \right) \right\|_{\frac{3}{2}, (\psi_i^k, \psi_{i+1}^k)}^2 \right)
\end{aligned}
$$

$$+ \sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\left(\left\|\left[\left(u_{i,j}^k\right)_{\theta_k}\right]\left(\tau_k,\psi_{i+1}^k\right)\right\|^2_{\frac{1}{2},(\eta_j^k,\eta_{j+1}^k)} + \left\|\left[\left(u_{i,j}^k\right)\right]\left(\tau_k,\psi_{i+1}^k\right)\right\|^2_{\frac{3}{2},(\eta_j^k,\eta_{j+1}^k)}\right)$$

$$+ \sum_{j=2}^{N}\left(\left\|\left(u_{1,j}^k\right)\left(\tau_k,\psi_1^k\right)\right\|^2_{\frac{3}{2},(\eta_j^k,\eta_{j+1}^k)} + \left\|\left(u_{I,j}^k\right)\left(\tau_k,\psi_{I+1}^k\right)\right\|^2_{\frac{3}{2},(\eta_j^k,\eta_{j+1}^k)}\right)$$

$$+ \sum_{i=1}^{I}\left(\left\|\left(u_{i,N}^k\right)(\ln\rho,\theta_k)\right\|^2_{\frac{3}{2},(\psi_i^k,\psi_{i+1}^k)}\right)\Bigg\}.$$

Adding a weighted combination of (2.22) and (2.23) and summing over i and j gives

$$\begin{aligned}
(2.25)\qquad & \sum_{j=1}^{N}\sum_{i=1}^{I_k}\Bigg\{\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(\left(u_{i,j}^k\right)^2_{\tau_k\tau_k} + 2\left(u_{i,j}^k\right)^2_{\tau_k\theta_k} + \left(u_{i,j}^k\right)^2_{\theta_k\theta_k}\right)d\tau_k d\theta_k \\
& + R\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(\left(\left(u_{i,j}^k\right)_{\tau_k}\right)^2 + \left(\left(u_{i,j}^k\right)_{\theta_k}\right)^2\right)d\tau_k d\theta_k\Bigg\} \\
& \leq (I) + (II) + (III) + (IV) + (V) + (VI) + (VII) + (VIII) + (IX).
\end{aligned}$$

Here the terms indicated by Roman numerals given above are as follows:

$$\begin{aligned}
(2.26)\qquad (I) &= \sum_{j=1}^{N}\sum_{i=1}^{I_k}\Bigg(\frac{KR}{2}\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{i,j}^k\right)^2 d\tau_k d\theta_k \\
&+ \left(1+\frac{R}{2K}\right)\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(\left(u_{i,j}^k\right)_{\tau_k\tau_k} + \left(u_{i,j}^k\right)_{\theta_k\theta_k}\right)^2 d\tau_k d\theta_k\Bigg), \\
(II) &= \sum_{j=1}^{N-1}\sum_{i=1}^{I_k}\left(\int_{\psi_i^k}^{\psi_{i+1}^k} -2\left[\left(u_{i,j}^k\right)_{\theta_k}\left(u_{i,j}^k\right)_{\tau_k\theta_k}\right]\left(\eta_{j+1}^k,\theta_k\right)d\theta_k\right), \\
(III) &= \sum_{i=1}^{I_k} 2\int_{\psi_i^k}^{\psi_{i+1}^k}\left(u_{i,N}^k\right)_{\theta_k}\left(u_{i,N}^k\right)_{\tau_k\theta_k}(\ln\rho,\theta_k)\,d\theta_k, \\
(IV) &= \sum_{i=1}^{I_k-1}\sum_{j=2}^{N}\left(\int_{\eta_j^k}^{\eta_{j+1}^k} 2\left[\left(u_{i,j}^k\right)_{\theta_k}\left(u_{i,j}^k\right)_{\tau_k\tau_k}\right]\left(\tau_k,\psi_{i+1}^k\right)d\tau_k\right), \\
(V) &= \sum_{j=1}^{N}\Bigg(2\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{1,j}^k\right)_{\theta_k}\left(u_{1,j}^k\right)_{\tau_k\tau_k}\left(\tau_k,\psi_1^k\right)d\tau_k \\
&- 2\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{I_k,j}^k\right)_{\theta_k}\left(u_{I_k,j}^k\right)_{\tau_k\tau_k}\left(\tau_k,\psi_{I_k+1}^k\right)d\tau_k\Bigg), \\
(VI) &= -R\sum_{j=1}^{N-1}\sum_{i=1}^{I_k}\left(\int_{\psi_i^k}^{\psi_{i+1}^k}\left[\left(u_{i,j}^k\right)\left(u_{i,j}^k\right)_{\tau_k}\right]\left(\eta_{j+1}^k,\theta_k\right)d\theta_k\right),
\end{aligned}$$

$$
\begin{aligned}
(VII) &= R\sum_{i=1}^{I_k}\int_{\psi_i^k}^{\psi_{i+1}^k}\left(u_{i,N}^k\right)\left(u_{i,N}^k\right)_{\tau_k}(\ln\rho,\theta_k)\,d\theta_k,\\
(VIII) &= R\sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\left(-\int_{\eta_j^k}^{\eta_{j+1}^k}\left[\left(u_{i,j}^k\right)\left(u_{i,j}^k\right)_{\theta_k}\right]\left(\tau_k,\psi_{i+1}^k\right)d\tau_k\right),\ \text{and}\\
(IX) &= R\sum_{j=1}^{N}\left(\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{I_k,j}^k\right)\left(u_{I_k,j}^k\right)_{\theta_k}\left(\tau_k,\psi_{I_k+1}^k\right)d\tau_k\right.\\
&\quad -\left.\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{1,j}^k\right)\left(u_{1,j}^k\right)_{\theta_k}\left(\tau_k,\psi_1^k\right)d\tau_k\right).
\end{aligned}
$$

Now using Lemma 2.4 we can conclude that

$$
\begin{aligned}
&\sum_{j=1}^{N}\sum_{i=1}^{I_k}\int_{\eta_j^k}^{\eta_{j+1}^k}\int_{\psi_i^k}^{\psi_{i+1}^k}\left(u_{i,j}^k\right)^2 d\tau_k d\theta_k \\
\leq\ & C\left(\sum_{j=1}^{N}\sum_{i=1}^{I_k}\int_{\eta_j^k}^{\eta_{j+1}^k}\int_{\psi_i^k}^{\psi_{i+1}^k}\left(\left(u_{i,j}^k\right)_{\theta_k}\right)^2 d\tau_k d\theta_k\right.\\
+\ & \sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\int_{\eta_j^k}^{\eta_{j+1}^k}\left[\left(u_{i,j}^k\right)\right]^2\left(\tau_k,\psi_{i+1}^k\right)d\tau_k\\
+\ & \left.\sum_{j=1}^{N}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{I_k,j}^k\right)^2\left(\tau_k,\psi_{I_k+1}^k\right)d\tau_k+\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{1,j}^k\right)^2\left(\tau_k,\psi_1^k\right)d\tau_k\right),
\end{aligned}
\tag{2.27}
$$

where

$$
C = 2\max_k\left\{\max\left(\frac{\left(\psi_{I_k+1}^k-\psi_1^k\right)^2}{2},\left(I_k+1\right)\left(\psi_{I_k+1}^k-\psi_1^k\right)\right)\right\}.
$$

Choosing R large enough and adding

$$
\begin{aligned}
2C&\left(\sum_{j=1}^{N}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{I_k,j}^k\right)^2\left(\tau_k,\psi_{I_k+1}^k\right)d\tau_k+\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{1,j}^k\right)^2\left(\tau_k,\psi_1^k\right)d\tau_k\right.\\
&\left.+\sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\int_{\eta_j^k}^{\eta_{j+1}^k}\left[\left(u_{i,j}^k\right)\right]^2\left(\tau_k,\psi_{i+1}^k\right)d\tau_k\right)
\end{aligned}
$$

to both sides of (2.25) and then applying (2.27) and choosing K small enough we get

the inequality

$$
\begin{aligned}
(2.28)\quad & \sum_{j=1}^{N}\sum_{i=1}^{I_k}\left\{\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(\left(\left(u_{i,j}^k\right)_{\tau_k\tau_k}\right)^2+2\left(\left(u_{i,j}^k\right)_{\tau_k\theta_k}\right)^2+\left(\left(u_{i,j}^k\right)_{\theta_k\theta_k}\right)^2\right)d\tau_k d\theta_k\right.\\
+ & \left.\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(\left(u_{i,j}^k\right)_{\tau_k}^2+\left(u_{i,j}^k\right)_{\theta_k}^2\right)d\tau_k d\theta_k\right\}+\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{i,j}^k\right)^2 d\tau_k d\theta_k\\
\le\ & T\sum_{j=1}^{N}\sum_{i=1}^{I_k}\int_{\psi_i^k}^{\psi_{i+1}^k}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(\left(u_{i,j}^k\right)_{\tau_k\tau_k}+\left(u_{i,j}^k\right)_{\theta_k\theta_k}\right)^2 d\tau_k d\theta_k\\
+ & 2C\left(\sum_{j=1}^{N}\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{I_k,j}^k\right)^2\left(\tau_k,\psi_{I_k+1}^k\right)d\tau_k+\int_{\eta_j^k}^{\eta_{j+1}^k}\left(u_{1,j}^k\right)^2\left(\tau_k,\psi_1^k\right)d\tau_k\right.\\
+ & \left.\sum_{i=1}^{I_k-1}\sum_{j=1}^{N}\int_{\eta_j^k}^{\eta_{j+1}^k}\left[\left(u_{i,j}^k\right)\right]^2\left(\tau_k,\psi_{i+1}^k\right)d\tau_k\right)\\
+ & (II)+(III)+(IV)+(V)+(VI)+(VII)+(VIII)+(IX).
\end{aligned}
$$

We shall now estimate the terms indicated by Roman numerals in the right hand side of (2.28) using Theorem 2.4. We begin by estimating

$$
\begin{aligned}
|(II)| = & \left|\sum_{j=1}^{N-1}\sum_{i=1}^{I_k}\left(-\int_{\psi_i^k}^{\psi_{i+1}^k}2\left[\left(u_{i,j}^k\right)_{\theta_k}\right]\left[\left(u_{i,j}^k\right)_{\tau_k\theta_k}\right]\left(\eta_{j+1}^k,\theta_k\right)d\theta_k\right.\right.\\
- & \int_{\psi_i^k}^{\psi_{i+1}^k}2\left[\left(u_{i,j}^k\right)_{\theta_k}\right]\left(u_{i,j}^k\right)_{\tau_k\theta_k}\left(\eta_{j+1}^k,\theta_k\right)d\theta_k\\
- & \left.\left.\int_{\psi_i^k}^{\psi_{i+1}^k}2\left(u_{i,j}^k\right)_{\theta_k}\left[\left(u_{i,j}^k\right)_{\tau_k\theta_k}\right]\left(\eta_{j+1}^k,\theta_k\right)d\theta_k\right)\right|.
\end{aligned}
$$

Now by Theorem 2.4

$$
\begin{aligned}
(2.29)\quad & \left|\int_{\psi_i^k}^{\psi_{i+1}^k}2\left[\left(u_{i,j}^k\right)_{\theta_k}\right]\left(u_{i,j}^k\right)_{\tau_k\theta_k}\left(\eta_{j+1}^k,\theta_k\right)d\theta_k\right|\\
\le\ & 2C(\ln N)\left\|\left[\left(u_{i,j}^k\right)_{\theta_k}\right]\left(\eta_{j+1}^k,\theta_k\right)\right\|_{\frac12,(\psi_i^k,\psi_{i+1}^k)}\times\left\|\left(u_{i,j}^k\right)_{\tau_k}\left(\eta_{j+1}^k,\theta_k\right)\right\|_{\frac12,(\psi_i^k,\psi_{i+1}^k)}\\
\le\ & \frac{(C\ln N)^2}{K}\left\|\left[\left(u_{i,j}^k\right)_{\theta_k}\right]\left(\eta_{j+1}^k,\theta_k\right)\right\|^2_{\frac12,(\psi_i^k,\psi_{i+1}^k)}+K\left\|\left(u_{i,j}^k\right)_{\tau_k}\left(\eta_{j+1}^k,\theta_k\right)\right\|^2_{\frac12,(\psi_i^k,\psi_{i+1}^k)}
\end{aligned}
$$

for any positive K.

By the trace theorem for Sobolev spaces there exists a constant M, such that

$$\left\| \left(u_{i,j}^k\right)_{\tau_k} \left(\eta_{j+1}^k, \theta_k\right) \right\|_{\frac{1}{2},\left(\psi_i^k,\psi_{i+1}^k\right)}^2 \leq M \left\| \left(u_{i,j}^k\right) \left(\tau_k, \theta_k\right) \right\|_{2,\tilde{\Omega}_{i,j}^k}^2 .$$

Choosing $K = \frac{1}{32M}$ we have

$$\begin{aligned}(2.30) \qquad & \left| \int_{\psi_i^k}^{\psi_{i+1}^k} 2 \left[\left(u_{i,j}^k\right)_{\theta_k} \right] \left(u_{i,j}^k\right)_{\tau_k \theta_k} \left(\eta_{j+1}^k, \theta_k\right) d\theta_k \right| \\ \leq \; & T \left(\ln N\right)^2 \left\| \left[\left(u_{i,j}^k\right)_{\theta_k} \right] \left(\eta_{j+1}^k, \theta_k\right) \right\|_{\frac{1}{2},\left(\psi_i^k,\psi_{i+1}^k\right)}^2 + \frac{1}{32} \left\| \left(u_{i,j}^k\right) \left(\tau_k, \theta_k\right) \right\|_{2,\tilde{\Omega}_{i,j}^k}^2 .\end{aligned}$$

And so we conclude that

$$\begin{aligned}(2.31) \qquad |(II)| \; \leq \; & C \left(\ln N\right)^2 \sum_{j=1}^{N-1} \sum_{i=1}^{I_k} \left(\left\| \left[\left(u_{i,j}^k\right)_{\tau_k} \right] \left(\eta_{j+1}^k, \theta_k\right) \right\|_{\frac{1}{2},\left(\psi_i^k,\psi_{i+1}^k\right)}^2 \right. \\ + \; & \left. \left\| \left[\left(u_{i,j}^k\right)_{\theta_k} \right] \left(\eta_{j+1}^k, \theta_k\right) \right\|_{\frac{1}{2},\left(\psi_i^k,\psi_{i+1}^k\right)}^2 \right) \\ + \; & \sum_{j=1}^{N-1} \sum_{i=1}^{I_k} \frac{1}{16} \left\| \left(u_{i,j}^k\right) \left(\tau_k, \theta_k\right) \right\|_{2,\tilde{\Omega}_{i,j}^k}^2 .\end{aligned}$$

Similarly we have

$$\begin{aligned}(2.32) \qquad |(V)| \; \leq \; & C \left(\ln N\right)^2 \sum_{j=2}^{N} \left(\left\| \left(u_{1,j}^k\right)_{\tau_k} \left(\tau_k, \psi_1^k\right) \right\|_{\frac{1}{2},\left(\eta_j^k,\eta_{j+1}^k\right)}^2 \right. \\ + \; & \left. \left\| \left(u_{I_k,j}^k\right)_{\tau_k} \left(\tau_k, \psi_{I_k+1}^k\right) \right\|_{\frac{1}{2},\left(\eta_j^k,\eta_{j+1}^k\right)}^2 \right) \\ + \; & \sum_{j=1}^{N} \frac{1}{16} \left(\left\| \left(u_{1,j}^k\right) \left(\tau_k, \theta_k\right) \right\|_{2,\tilde{\Omega}_{1,j}^k}^2 + \left\| \left(u_{I_k,j}^k\right) \left(\tau_k, \theta_k\right) \right\|_{2,\tilde{\Omega}_{I_k,j}^k}^2 \right) .\end{aligned}$$

We can estimate the terms (IV), (VI), $(VIII)$ and (IX) in the right hand side of (2.26) in a similar manner. Putting all these estimates together we can write (2.28) in the form

$$\begin{aligned}(2.33) \qquad & \frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{I_k} \left\| \left(u_{i,j}^k\right) \left(\tau_k, \theta_k\right) \right\|_{2,\tilde{\Omega}_{i,j}^k}^2 \\ \leq \; & C \left(\ln N\right)^2 \left\{ \sum_{j=1}^{N} \sum_{i=1}^{I_k} \int_{\psi_i^k}^{\psi_{i+1}^k} \int_{\eta_j^k}^{\eta_{j+1}^k} \left(\Delta u_{i,j}^k\right)^2 d\tau_k d\theta_k \right.\end{aligned}$$

$$+ \sum_{j=1}^{N-1}\sum_{i=1}^{I_k}\left(\left\|\left[\left(u_{i,j}^k\right)_{\tau_k}\right]\left(\eta_{j+1}^k,\theta_k\right)\right\|^2_{\frac{1}{2},\left(\psi_i^k,\psi_{i+1}^k\right)} + \left\|\left[\left(u_{i,j}^k\right)\right]\left(\eta_{j+1}^k,\theta_k\right)\right\|^2_{\frac{3}{2},\left(\psi_i^k,\psi_{i+1}^k\right)}\right)$$

$$+ \sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\left(\left\|\left[\left(u_{i,j}^k\right)\right]\left(\tau_k,\psi_{i+1}^k\right)\right\|^2_{\frac{3}{2},\left(\eta_j^k,\eta_{j+1}^k\right)} + \left\|\left[\left(u_{i,j}^k\right)_{\theta_k}\right]\left(\tau_k,\psi_{i+1}^k\right)\right\|^2_{\frac{1}{2},\left(\eta_j^k,\eta_{j+1}^k\right)}\right)$$

$$\left. + \sum_{j=1}^{N}\left(\left\|\left(u_{1,j}^k\right)\left(\tau_k,\psi_1^k\right)\right\|^2_{\frac{3}{2},\left(\eta_j^k,\eta_{j+1}^k\right)} + \left\|\left(u_{I_k,j}^k\right)\left(\tau_k,\psi_{I_k+1}^k\right)\right\|^2_{\frac{3}{2},\left(\eta_j^k,\eta_{j+1}^k\right)}\right)\right\}$$

$$+ \sum_{i=1}^{I_k}\left(R\int_{\psi_i^k}^{\psi_{i+1}^k}\left(u_{i,N}^k\right)\left(u_{i,N}^k\right)_{\tau_k}(\ln\rho,\theta_k)\,d\theta_k\right.$$

$$\left. + \; 2\int_{\psi_i^k}^{\psi_{i+1}^k}\left(u_{i,N}^k\right)_{\theta_k}\left(u_{i,N}^k\right)_{\tau_k\theta_k}(\ln\rho,\theta_k)\,d\theta_k\right).$$

Estimating the last two terms in the same way we get the result.□

We shall formulate (2.33) in a more compact form. Recollect that B_ρ^k denotes the circular arc with center A_k and radius ρ. Let $\widetilde{B}_\rho^k$ be the representation of this curve in τ_k,θ_k coordinates, i.e. $\widetilde{B}_\rho^k = \left\{(\tau_k,\theta_k)\,\middle|\,\tau_k = \ln\rho,\ \psi_l^k < \theta_k < \psi_u^k\right\}$.
Similarly let $\widetilde{\Gamma}_k$ $\widetilde{\Gamma}_{k+1}$ be the representation of Γ_k and Γ_{k+1} in (τ_k,θ_k) coordinates. Let γ_l be a side of $\Omega_{i,j}^k$ and let $\widetilde{\gamma}_l$ be its representation in (τ_k,θ_k) coordinates. We then have the following representation of (2.33).
Consider the function $\left\{u_{i,j}^k\right\}_{i,j}$ defined on $\widetilde{\Omega}_{i,j}^k \subseteq \widetilde{\Omega}^k$. Then the inequality

$$\frac{1}{2}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\left(u_{i,j}^k\right)(\tau_k,\theta_k)\right\|^2_{2,\widetilde{\Omega}_{i,j}^k} \tag{2.34}$$

$$\leq C(\ln N)^2\left\{\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\Delta u_{i,j}^k(\tau_k,\theta_k)\right\|^2_{0,\widetilde{\Omega}_{i,j}^k}\right.$$

$$+ \sum_{m=k}^{k+1}\sum_{\widetilde{\gamma}_i\subseteq\widetilde{\Gamma}_m\bigcap\partial\widetilde{\Omega}^k}\left(\left\|u^k\right\|^2_{0,\widetilde{\gamma}_i} + \left\|u_{\tau_k}^k\right\|^2_{1/2,\widetilde{\gamma}_i}\right)$$

$$\left. + \sum_{\widetilde{\gamma}_i\subseteq\widetilde{\Omega}^k}\left(\left\|\left[u^k\right]\right\|^2_{0,\widetilde{\gamma}_i} + \left\|\left[u_{\tau_k}^k\right]\right\|^2_{1/2,\widetilde{\gamma}_i} + \left\|\left[u_{\theta_k}^k\right]\right\|^2_{1/2,\widetilde{\gamma}_i}\right)\right\}$$

$$+ \sum_{i=1}^{I_k}\left(R\int_{\psi_i^k}^{\psi_{i+1}^k}\left(u_{i,N}^k\right)\left(u_{i,N}^k\right)_{\tau_k}(\ln\rho,\theta_k)\,d\theta_k\right.$$

$$\left. + \; 2\int_{\psi_i^k}^{\psi_{i+1}^k}\left(u_{i,N}^k\right)_{\theta_k}\left(u_{i,N}^k\right)_{\tau_k\theta_k}(\ln\rho,\theta_k)\,d\theta_k\right)$$

is valid.

The estimate (2.34) follows immediately from (2.33).

We need to obtain a similar estimate for T^k. On each $\overline{\Omega}^k_{i,j} \subseteq \overline{T}^k$ a function $u^k_{i,j}(x,y)$ is defined. Now, this $\Omega^k_{i,j} \subseteq \Omega^{p+1}$ and hence $\Omega^k_{i,j} = \Omega^{p+1}_l$ for some l. We shall use $\Omega^k_{i,j}$ and Ω^{p+1}_l interchangeably in what follows.

We now need to obtain an energy inequality similar to Theorem 2.1 on T^k. Integrating by parts repeatedly we get

$$
\begin{aligned}
(2.35)\qquad & \rho^2 \int\!\!\int_{\mathcal{O}} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 dxdy \\
&= \rho^2 \left(\int\!\!\int_{\mathcal{O}} \left(\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 \right) dxdy \right. \\
&+ \left. 2\int_{\partial\mathcal{O}} \frac{\partial w}{\partial y}\frac{d}{ds}\left(\frac{\partial w}{\partial x}\right) ds \right).
\end{aligned}
$$

Here s denotes arc length along $\partial\mathcal{O}$ measured from some point on it and the line integral is evaluated in the clockwise direction and n denotes the outward normal to $\partial\mathcal{O}$, the boundary of $\mathcal{O}$. Hence

$$
\begin{aligned}
& \rho^2 \int_{\Omega^k_{i,N+1}}\!\!\int \left(\frac{\partial^2 u^k_{i,N+1}}{\partial x^2} + \frac{\partial^2 u^k_{i,N+1}}{\partial y^2}\right)^2 dxdy \\
&= \rho^2 \int_{\Omega^k_{i,N+1}}\!\!\int \left(\left(\frac{\partial^2 u^k_{i,N+1}}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u^k_{i,N+1}}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u^k_{i,N+1}}{\partial y^2}\right)^2 \right) dxdy \\
&+ 2\rho^2 \left(\int_{\partial\Omega^k_{i,N+1}\cap(B^k_\rho)^c} + \int_{\partial\Omega^k_{i,N+1}\cap(B^k_\rho)} \right) \frac{\partial u^k_{i,N+1}}{\partial y}\frac{d}{ds}\left(\frac{\partial u^k_{i,N+1}}{\partial x}\right) ds.
\end{aligned}
$$

Here B^k_ρ denotes the boundary of the circle of radius ρ with center at A_k. Now a simple calculation yields

$$
\begin{aligned}
(2.36)\qquad & 2\rho^2 \int_{\partial\Omega^k_{i,N+1}\cap B^k_\rho} \frac{\partial u^k_{i,N+1}}{\partial y}\frac{d}{ds}\left(\frac{\partial u^k_{i,N+1}}{\partial x}\right) ds \\
&= 2\int_{\psi^k_i}^{\psi^k_{i+1}} \left(u^k_{i,N+1}\right)_{\theta_k} \left(u^k_{i,N+1}\right)_{\theta_k\tau_k} (\ln\rho,\theta_k)\, d\theta_k \\
&- \int_{\psi^k_i}^{\psi^k_{i+1}} \left(\left(u^k_{i,N+1}\right)^2_{\tau_k} + \left(u^k_{i,N+1}\right)^2_{\theta_k} \right) (\ln\rho,\theta_k)\, d\theta_k \\
&+ \rho^2 B\left(\theta_k, \left(u^k_{i,N+1}\right)_{\tau_k}, \left(u^k_{i,N+1}\right)_{n_k}\right) (\ln\rho,\theta_k)\Big|_{\psi^k_i}^{\psi^k_{i+1}}.
\end{aligned}
$$

Here

$$
\begin{aligned}
(2.37) \qquad B(\theta, a, b) &= \left(a^2\frac{\sin 2\theta}{2} - b^2\frac{\sin 2\theta}{2} - 2ab\sin^2\theta\right), \text{ and} \\
\frac{\partial}{\partial n_k} &= \frac{1}{\rho}\frac{\partial}{\partial \theta_k}.
\end{aligned}
$$

Next suppose Ω_l^{p+1} is such that $\partial\Omega_l^{p+1} \bigcap \Gamma_m \neq \phi$. Then $\partial\Omega^{p+1} \bigcap \Gamma_m$ is the straight line joining the points D_j^m and D_{j+1}^m for $m \in \{k, k+1\}$ and for some $1 \leq j \leq M_m$ as shown in Fig. 2.4. Here $M_m = J_m - N$. Now we can show that

$$
\begin{aligned}
(2.38) \qquad & 2\rho^2 \int_{\partial\Omega_l^{p+1} \bigcap \Gamma_m} \frac{\partial u_l^{p+1}}{\partial y}\frac{d}{ds}\left(\frac{\partial u_l^{p+1}}{\partial x}\right) ds \\
= \; & 2\rho^2 \int_{\partial\Omega_l^{p+1} \bigcap \Gamma_m} \left(u_l^{p+1}\right)_{\nu_m} \left(u_l^{p+1}\right)_{\sigma_m\sigma_m} d\sigma_m \\
+ \; & \rho^2 B\left(\psi_1^m, \left(u_l^{p+1}\right)_{\sigma_m} \left(u_l^{p+1}\right)_{\nu_m}\right)\Big|_{D_j^m}^{D_{j+1}^m}.
\end{aligned}
$$

Here $\frac{d}{d\sigma_m}$ denotes the tangential derivative and $\frac{d}{d\nu_m}$ the normal derivative along Γ_m.

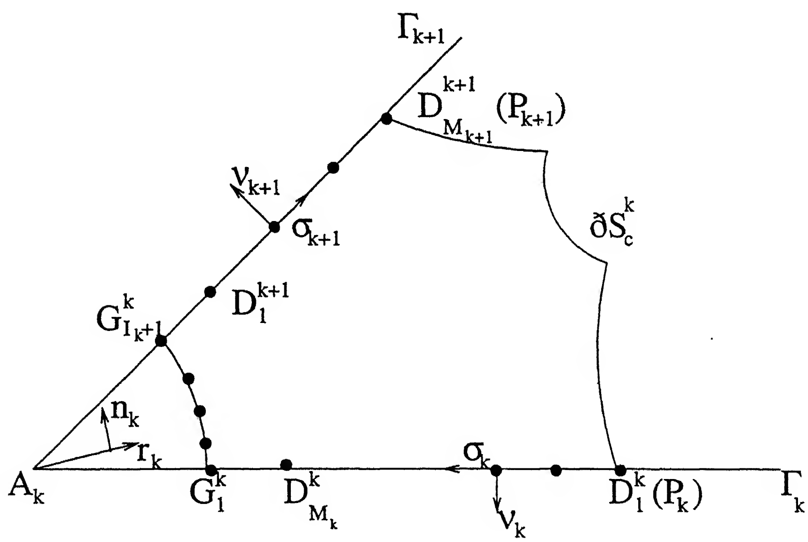

Figure 2.4: Points on the adjacent boundaries

Now

$$
\begin{aligned}
(2.39) \qquad & R\int_{\Omega_{i,N+1}^k}\int \left(\left(\frac{\partial u_{i,N+1}^k}{\partial x}\right)^2 + \left(\frac{\partial u_{i,N+1}^k}{\partial y}\right)^2\right) dxdy \\
\leq \; & \frac{R}{2K}\int_{\Omega_{i,N+1}^k}\int \left(\frac{\partial^2 u_{i,N+1}^k}{\partial x^2} + \frac{\partial^2 u_{i,N+1}^k}{\partial y^2}\right)^2 dxdy
\end{aligned}
$$

$$+ \frac{KR}{2} \int_{\Omega_{i,N+1}^k} \int \left(u_{i,N+1}^k\right)^2 dxdy - R \int_{\psi_i^k}^{\psi_{i+1}^k} \left(u_{i,N+1}^k\right) \left(u_{i,N+1}^k\right)_{\tau_k} (\ln \rho, \theta_k)\, d\theta_k$$
$$+ R \int_{\partial\Omega_{i,N+1}^k \cap (B_\rho^k)^c} \left(u_{i,N+1}^k\right) \frac{\partial \left(u_{i,N+1}^k\right)}{\partial n} ds.$$

Similarly for $\Omega_l^{p+1} = \Omega_{i,j}^k$ with $j > N+1$ we get

$$R \int_{\Omega_l^{p+1}} \int \left(\left(\frac{\partial u_l^{p+1}}{\partial x}\right)^2 + \left(\frac{\partial u_l^{p+1}}{\partial y}\right)^2 \right) dxdy \tag{2.40}$$
$$\leq \frac{R}{2K} \int_{\Omega_l^{p+1}} \int \left(\frac{\partial^2 u_l^{p+1}}{\partial x^2} + \frac{\partial^2 u_l^{p+1}}{\partial y^2} \right)^2 dxdy$$
$$+ \frac{KR}{2} \int_{\Omega_l^{p+1}} \int \left(u_l^{p+1}\right)^2 dxdy + R \int_{\partial\Omega_l^{p+1}} u_l^{p+1} \frac{\partial u_l^{p+1}}{\partial n} ds.$$

Let $L^k = \left\{ l : \Omega_l^{p+1} \subseteq T^k \right\}$. We can now prove the following lemma:

Lemma 2.1

$$\sum_{l \in L^k} \left(\rho^2 \int_{\Omega_l^{p+1}} \int \left(\left(\frac{\partial^2 u_l^{p+1}}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 u_l^{p+1}}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 u_l^{p+1}}{\partial y^2}\right)^2 \right) dxdy \right. \tag{2.41}$$
$$\left. + R \int_{\Omega_l^{p+1}} \int \left(\left(\frac{\partial u_l^{p+1}}{\partial x}\right)^2 + \left(\frac{\partial u_l^{p+1}}{\partial y}\right)^2 \right) dxdy \right)$$
$$\leq (I) + (II) + (III) + (IV) + (V) + (VI) + (VII)$$
$$+ (VIII) + (IX) + (X).$$

The terms indicated by Roman numerals are as follows:

$$(I) = \left(\rho^2 + \frac{R}{2K}\right) \sum_{l \in L^k} \int_{\Omega_l^{p+1}} \int \left(\Delta u_l^{p+1}(x,y)\right)^2 dxdy, \tag{2.42}$$
$$(II) = \frac{KR}{2} \sum_{l \in L^k} \int_{\Omega_l^{p+1}} \int \left(u_l^{p+1}(x,y)\right)^2 dxdy,$$
$$(III) = \sum_{i=1}^{I_k} \left\{ -2 \int_{\psi_i^k}^{\psi_{i+1}^k} \left(u_{i,N+1}^k\right)_{\theta_k} \left(u_{i,N+1}^k\right)_{\tau_k \theta_k} (\ln \rho, \theta_k)\, d\theta_k \right.$$
$$\left. - R \int_{\psi_i^k}^{\psi_{i+1}^k} \left(u_{i,N+1}^k\right) \left(u_{i,N+1}^k\right)_{\tau_k} (\ln \rho, \theta_k)\, d\theta_k \right\},$$
$$(IV) = \rho \sum_{i=1}^{I_k} \int_{\partial\Omega_{i,N+1}^k \cap B_\rho^k} \int \left(\left(u_{i,N+1}^k\right)_x^2 + \left(u_{i,N+1}^k\right)_y^2 \right) ds,$$

$$
\begin{aligned}
(V) &= R\sum_{l\in L^k}\Bigg\{\Bigg(\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap T^k} + \sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\partial S_c^k}\Bigg)\int_{\gamma_s} u_l^{p+1}\frac{\partial u_l^{p+1}}{\partial n}ds \\
&\quad + \sum_{m=k}^{k+1}\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\Gamma_m}\int_{\gamma_s}\left(u_l^{p+1}\right)\left(u_l^{p+1}\right)_{\nu_m} d\sigma_m\Bigg\}, \\
(VI) &= -2\rho^2\sum_{l\in L^k}\Bigg\{\Bigg(\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap T^k} + \sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\partial S_c^k}\Bigg)\int_{\gamma_s}\frac{\partial u_l^{p+1}}{\partial y}\frac{d}{ds}\left(\frac{\partial u_l^{p+1}}{\partial x}\right)ds \\
&\quad + \sum_{m=k}^{k+1}\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\Gamma_m}\int_{\gamma_s}\left(u_l^{p+1}\right)_{\nu_m}\left(u_l^{p+1}\right)_{\sigma_m\sigma_m} d\sigma_m\Bigg\}, \\
(VII) &= -\sum_{i=1}^{I_k}\rho^2 B\left(\theta_k, \left(u_{i,N+1}^k\right)_{r_k}, \left(u_{i,N+1}^k\right)_{n_k}\right)(\ln\rho,\theta_k)\Big|_{\psi_i^k}^{\psi_{i+1}^k}, \\
(VIII) &= -\sum_{m=k}^{k+1}\sum_{i=1}^{M_m-1}\sum_{l,\partial\Omega_l^{p+1}\bigcap\Gamma_{m,i}\neq\phi}\rho^2 B\left(\psi_1^m, \left(u_l^{p+1}\right)_{\sigma_m}, \left(u_l^{p+1}\right)_{\nu_m}\right)\Big|_{D_i^m}^{D_{i+1}^m}, \\
(IX) &= -\rho^2 B\left(\psi_1^k, \left(u_{1,N+1}^k\right)_{\sigma_k}, \left(u_{1,N+1}^k\right)_{\nu_k}\right)\Big|_{D_{M_k}^k}^{G_1^k}, \qquad \text{and} \\
(X) &= -\rho^2 B\left(\psi_1^{k+1}, \left(u_{I_k,N+1}^k\right)_{\sigma_{k+1}}, \left(u_{I_k,N+1}^k\right)_{\nu_{k+1}}\right)\Big|_{G_{I_k+1}^k}^{D_1^{k+1}}.
\end{aligned}
$$

By γ_s we denote an arc which is a side of $\partial\Omega_l^{p+1}$ for $l \in L^k$. Here $\frac{\partial}{\partial r_k}$ denotes the radial derivative and $\frac{\partial}{\partial n_k}$ the tangential derivative to the circle with center at A_k and radius ρ, i.e. $\frac{\partial}{\partial n_k} = \frac{1}{\rho}\frac{\partial}{\partial\theta_k}$. Moreover $\frac{\partial}{\partial\sigma_k}$ denotes the tangential derivative and $\frac{\partial}{\partial\nu_k}$ the normal derivative to the side Γ_k (Fig. 2.4). Finally $\Gamma_{m,i}$ is the open subset of the straight line Γ_m between the points D_i^m and D_{i+1}^m and $u_{I_{m+1},N+1}^{m+1}$ denotes $u_{I_1,N+1}^1$ if $m = p$.

Using the estimates (2.36 - 2.40) we obtain (2.41).□

Recall from (2.8) that there exists a mapping M_l^{p+1} from the unit square S to $\overline{\Omega}_l^{p+1}$ given by

$$
\begin{aligned}
x &= X_l^{p+1}(\xi,\eta) \\
y &= Y_l^{p+1}(\xi,\eta)
\end{aligned}
$$

Similarly there exists a mapping $M_{i,N+1}^k$ from S to $\overline{\Omega}_{i,N+1}^k$.

We now define another semi norm in terms of the transformed variables ξ and η:

$$|v(\xi,\eta)|^2_{m,S} = \sum_{|\alpha|=m} \int_S\int \left|D^{\alpha_1}_\xi D^{\alpha_2}_\eta v\left(X^{p+1}_l(\xi,\eta), Y^{p+1}_l(\xi,\eta)\right)\right|^2 d\xi d\eta$$

Let

$$\text{(2.43a)} \quad \left|M^{p+1}_l\right|_{m,\infty,S} = \operatorname{ess}\sup_{(\xi,\eta)\in S}\left(\max\left(\max_{|\alpha|\le m}\left(\left|D^\alpha X^{p+1}_l\right|\right), \max_{|\alpha|\le m}\left(\left|D^\alpha Y^{p+1}_l\right|\right)\right)\right).$$

Then we have the following results [15]

$$\begin{aligned}
\text{(2.43b)}\ |u(\xi,\eta)|^2_{0,S} &\le \frac{C}{\left|J_{M^{p+1}_l}\right|}|u|^2_{0,\Omega^{p+1}_l} \le C|u|^2_{0,\Omega^{p+1}_l},\\
\text{(2.43c)}\ |u(\xi,\eta)|^2_{1,S} &\le C\frac{\left|M^{p+1}_l\right|^2_{1,\infty,S}}{\left|J_{M^{p+1}_l}\right|}|u|^2_{1,\Omega^{p+1}_l} \le C|u|^2_{1,\Omega^{p+1}_l}, \text{ and}\\
\text{(2.43d)}\ |u(\xi,\eta)|^2_{2,S} &\le \frac{C}{\left|J_{M^{p+1}_l}\right|}\left(\left|M^{p+1}_l\right|^4_{1,\infty,S}|u|^2_{2,\Omega^{p+1}_l} + \left|M^{p+1}_l\right|^2_{2,\infty,S}|u|^2_{1,\Omega^{p+1}_l}\right)\\
&\le C\left(|u|^2_{2,\Omega^{p+1}_l} + |u|^2_{1,\Omega^{p+1}_l}\right).
\end{aligned}$$

Here $J_{M^{p+1}_l}$ denotes the Jacobian of the mapping M^{p+1}_l as defined in (2.9a - 2.9b) and

$$\left|J_{M^{p+1}_l}\right| = \min_{(\xi,\eta)\in S}\left|J_{M^{p+1}_l}(\xi,\eta)\right|;$$

note that we have used the bound given in (2.9b) to arrive at the above results.

Consider the point D^k_i in Fig. 2.4. Then there exist two domains Ω^{p+1}_l and Ω^{p+1}_m on whose boundary D^k_i lies. Let

$$\left[w^{p+1}\right]\left(D^k_i\right) = w^{p+1}_m\left(D^k_i\right) - w^{p+1}_l\left(D^k_i\right),$$

where $\partial\Omega^{p+1}_m \bigcap \Gamma_k$ is traversed first if we travel along Γ_k from D^k_1 to $D^k_{M_k}$.

Moreover let

$$\left[w^{p+1}\right]\left(G^k_i\right) = w^k_{i,N+1}\left(G^k_i\right) - w^k_{i-1,N+1}\left(G^k_i\right).$$

We now prove the following lemma.

Lemma 2.2

$$|(VII)+(VIII)+(IX)+(X)| \le (XI)+\frac{6\rho^2}{32}\sum_{l\in L^k}\left(\left|u_l^{p+1}\right|^2_{1,\Omega_l^{p+1}}+\left|u_l^{p+1}\right|^2_{2,\Omega_l^{p+1}}\right), \tag{2.44}$$

where

$$\begin{aligned}(XI) &= C\ln N\left(\sum_{i=2}^{I_k}\left|\left[(u^{p+1})_x\right](G_i^k)\right|^2+\left|\left[(u^{p+1})_y\right](G_i^k)\right|^2\right.\\ &+ \left.\sum_{m=k}^{k+1}\sum_{i=2-\delta_{m,k+1}}^{M_m-\delta_{m,k+1}}\left(\left(\left[(u^{p+1})_x\right](D_i^m)\right)^2+\left(\left[(u^{p+1})_y\right](D_i^m)\right)^2\right)\right)\\ &+ \sum_{m=k}^{k+1}(-1)^{m+k-1}\rho^2\left\{B\left(\psi_1^m,u_{\nu_m},u_{\sigma_m}\right)\right\}(P_m).\end{aligned} \tag{2.45}$$

Here P_m is D_1^k if $m=k$ and P_m is $D_{M_{k+1}}^{k+1}$ if $m=k+1$.

We first estimate one of the terms in the right hand side of (2.44). Now

$$\begin{aligned}&\left|\rho^2\sin^2\left(\psi_i^k\right)\left\{\left(u_{i-1,N+1}^k\right)_{r_k}\left(u_{i-1,N+1}^k\right)_{n_k}-\left(u_{i,N+1}^k\right)_{r_k}\left(u_{i,N+1}^k\right)_{n_k}\right\}(G_i^k)\right|\\ \le\ &\rho^2\left(\left|\left(u_{i,N+1}^k\right)_{r_k}(G_i^k)\right|\left|\left[\left(u_{i,N+1}^k\right)_{n_k}\right](G_i^k)\right|\right.\\ +\ &\left.\left|\left(u_{i-1,N+1}^k\right)_{n_k}(G_i^k)\right|\left|\left[\left(u_{i,N+1}^k\right)_{r_k}\right](G_i^k)\right|\right).\end{aligned} \tag{2.46}$$

Now by Corollary 4.80 of [42] we have that if a and b are real numbers such that $a^2+b^2=1$ and w is a smooth function defined on Ω_l^{p+1} such that

$$w\left(X_l^{p+1}(\xi,\eta),Y_l^{p+1}(\xi,\eta)\right)=\sum_{n=0}^{N}\sum_{m=0}^{N}a_{m,n}\xi^m\eta^n$$

then

$$|(aw_x+bw_y)(P)|^2\le C(\ln N)\left(|w|^2_{1,\Omega_l^{p+1}}+|w|^2_{2,\Omega_l^{p+1}}\right). \tag{2.47}$$

Using (2.47) we obtain

$$
\begin{aligned}
& \left|\rho^2 \sin^2\left(\psi_i^k\right)\left\{\left(u_{i-1,N+1}^k\right)_{r_k}\left(u_{i-1,N+1}^k\right)_{n_k}-\left(u_{i,N+1}^k\right)_{r_k}\left(u_{i,N+1}^k\right)_{n_k}\right\}\left(G_i^k\right)\right| \\
\leq\ & C \ln N\left\{\left(\left[\left(u_{i,N+1}^k\right)_{r_k}\right]\left(G_i^k\right)\right)^2+\left(\left[\left(u_{i,N+1}^k\right)_{n_k}\right]\left(G_i^k\right)\right)^2\right\} \\
+\ & \frac{\rho^2}{32}\left(\left|u_{i,N+1}^k\right|_{1,\Omega_{i,N+1}^k}^2+\left|u_{i,N+1}^k\right|_{2,\Omega_{i,N+1}^k}^2+\left|u_{i-1,N+1}^k\right|_{1,\Omega_{i-1,N+1}^k}^2+\left|u_{i-1,N+1}^k\right|_{2,\Omega_{i-1,N+1}^k}^2\right).
\end{aligned}
$$

Treating the other terms in the same way we obtain the result.□

We now estimate the term (IV) in (2.42). Let w be a smooth function on $\Omega_{i,N+1}^k$. Then

$$
\begin{aligned}
\int_{\partial\Omega_{i,N+1}^k \cap B_\rho^k} w^2 ds &= \int_{\psi_i^k}^{\psi_{i+1}^k} w^2\left(\rho, \theta_k\right) \rho d\theta_k \\
&= \int_{\psi_i^k}^{\psi_{i+1}^k} \int_\rho^\nu -\rho \frac{\partial}{\partial r_k}\left(\frac{\nu-r_k}{\nu-\rho} w^2\right) dr_k d\theta_k \\
&= \int_{\psi_i^k}^{\psi_{i+1}^k} \int_\rho^\nu \frac{\rho}{\nu-\rho} w^2 dr_k d\theta_k+\int_{\psi_i^k}^{\psi_{i+1}^k} \int_\rho^\nu -2\rho\left(\frac{\nu-r_k}{\nu-\rho}\right) w w_{r_k} dr_k d\theta_k \\
&\leq \frac{1}{\nu-\rho} \int_{\psi_i^k}^{\psi_{i+1}^k} \int_\rho^\nu w^2 r_k dr_k d\theta_k+2 \int_{\psi_i^k}^{\psi_{i+1}^k} \int_\rho^\nu\left|w w_{r_k}\right| r_k dr_k d\theta_k.
\end{aligned}
$$

And so we obtain

$$
\begin{aligned}
& \int_{B_\rho^k \cap \partial\Omega_{i,N+1}^k} w^2 ds \\
& \leq\left(\frac{1}{\nu-\rho}+\alpha\right) \int_{\psi_i^k}^{\psi_{i+1}^k} \int_\rho^\nu w^2 r_k dr_k d\theta_k+\frac{1}{\alpha} \int_{\psi_i^k}^{\psi_{i+1}^k} \int_\rho^\nu\left(w_{r_k}\right)^2 r_k dr_k d\theta_k .
\end{aligned} \tag{2.48}
$$

for any $\alpha>0$.

Hence using (2.48) we get

$$
\begin{aligned}
(IV) &= \rho \int_{\partial\Omega_{i,N+1}^k \cap B_\rho^k}\left(\left(u_{i,N+1}^k\right)_x^2+\left(u_{i,N+1}^k\right)_y^2\right) ds \\
&\leq\left(\frac{\rho}{\nu-\rho}+\alpha\rho\right) \int_{\Omega_{i,N+1}^k} \int\left(\left(u_{i,N+1}^k\right)_x^2+\left(u_{i,N+1}^k\right)_y^2\right) dx dy \\
&+\frac{\rho}{\alpha} \int_{\Omega_{i,N+1}^k} \int\left(\left(u_{i,N+1}^k\right)_{xx}^2+2\left(u_{i,N+1}^k\right)_{xy}^2+\left(u_{i,N+1}^k\right)_{yy}^2\right) dx dy .
\end{aligned} \tag{2.49}
$$

Choose α so large that $\frac{\rho}{\alpha} \leq \frac{\rho^2}{32}$ and choose $R > \frac{\rho}{\nu-\rho} + \alpha\rho + \frac{\rho^2}{2}$. Then combining (2.49) with Lemma 2.2, we have the result

$$\sum_{l\in L^k} \frac{25}{32}\rho^2 \left|u_l^{p+1}\right|^2_{2,\Omega_l^{p+1}} + \sum_{l\in L^k}\left(R - \frac{\rho}{\nu-\rho} - \alpha\rho\right)\left|u_l^{p+1}\right|^2_{1,\Omega_l^{p+1}} \leq ((I)+(II)+(III)+(V)+(VI))+(XI). \tag{2.50}$$

We now obtain an estimate for the term (VI) as defined in (2.42).

We shall estimate the first term in (VI). Now

$$\begin{aligned}&\left|2\rho^2 \sum_{l\in L^k} \sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap T^k} \int_{\gamma_s} \frac{\partial u_l^{p+1}}{\partial y}\frac{d}{ds}\left(\frac{\partial u_l^{p+1}}{\partial x}\right) ds\right| \\ &\leq 2\rho^2 \sum_{\gamma_s\subseteq T^k} \left|\int_{\gamma_s}\left[\frac{\partial u_l^{p+1}}{\partial y}\frac{d}{ds}\left(\frac{\partial u_l^{p+1}}{\partial x}\right)\right] ds\right|.\end{aligned} \tag{2.51}$$

Let us get an upper bound on a typical element in the sum, in the right hand side of the above inequality, which is of the form

$$2\rho^2\left|\int_{\gamma_s}\left(\left(\frac{\partial u_m^{p+1}}{\partial y}\right)\frac{d}{ds}\left(\frac{\partial u_m^{p+1}}{\partial x}\right) - \left(\frac{\partial u_n^{p+1}}{\partial y}\right)\frac{d}{ds}\left(\frac{\partial u_n^{p+1}}{\partial x}\right)\right) ds\right|. \tag{2.52}$$

We shall assume, to be specific, that $\partial\Omega_{m,l}^{p+1}$ is the image of the side $\xi = 1$ of the square S under the mapping M_m^{p+1} and $\partial\Omega_{n,j}^{p+1}$ is the image of the side $\xi = 0$ of S under the mapping M_n^{p+1}. Recall from (2.16 - 2.17) that there exists i, j and k such that $\Omega_l^{p+1} = \Omega_{i,j}^k$ for some i, j with $j > N$. Hence the mapping M_m^{p+1} is the mapping $M_{i,j}^k$ from S to $\overline{\Omega}_{i,j}^k$ as in (2.8). This representation is needed only for $1 \leq k \leq p, N < j \leq J_k$ and $1 \leq i \leq I_{k,j}$. Now $J_k = N + O(1)$ and $I_{k,j} \leq I$ and hence there are a fixed number of $\Omega_{i,j}^k$ for which this representation is needed even if we let $N \to \infty$. As such we may assume

$$\max_{i,j,k,j>N} \left|M_{i,j}^k\right|_{m,\infty,S} \leq C_m \tag{2.53}$$

where the norm has been defined in (2.43a). Note that C_m is independent of N. We

shall impose further restrictions on C_m in the second part of this paper where we shall examine the accuracy of our numerical scheme. Here we shall only establish the stability of our scheme and for that an estimate of the type (2.53) is adequate.

Now

$$\text{(2.54a)} \qquad \frac{\partial u_m^{p+1}}{\partial x} = \left(u_m^{p+1}\right)_\xi \xi_x + \left(u_m^{p+1}\right)_\eta \eta_x, \text{ and}$$

$$\text{(2.54b)} \qquad \frac{\partial u_m^{p+1}}{\partial y} = \left(u_m^{p+1}\right)_\xi \xi_y + \left(u_m^{p+1}\right)_\eta \eta_y.$$

We have that

$$\begin{cases} x = X_m^{p+1}(\xi,\eta) \\ y = Y_m^{p+1}(\xi,\eta) \end{cases}, \; 0 \le \xi \le 1, \; 0 \le \eta \le 1.$$

Let $\widehat{\xi_x}(\xi,\eta), \widehat{\eta_x}(\xi,\eta), \widehat{\xi_y}(\xi,\eta)$ and $\widehat{\eta_y}(\xi,\eta)$ be the unique polynomials in ξ and η which are the orthogonal projections of $\xi_x(\xi,\eta), \eta_x(\xi,\eta), \xi_y(\xi,\eta)$ and $\eta_y(\xi,\eta)$ into the space of polynomials of degree $(N-1)$ in each variable separately with respect to the usual inner product in $H^2\left((0,1)^2\right)$, as defined in [14]. We now define approximations to the derivatives $\frac{\partial u_m^{p+1}}{\partial x}$ and $\frac{\partial u_m^{p+1}}{\partial y}$ as follows. Let

$$\text{(2.55a)} \qquad \left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a = \left(u_m^{p+1}\right)_\xi \widehat{\xi_x} + \left(u_m^{p+1}\right)_\eta \widehat{\eta_x}, \text{ and}$$

$$\text{(2.55b)} \qquad \left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a = \left(u_m^{p+1}\right)_\xi \widehat{\xi_y} + \left(u_m^{p+1}\right)_\eta \widehat{\eta_y}.$$

Using the approximation results in [14] we have that

$$\text{(2.56)} \qquad \left|\xi_x - \widehat{\xi_x}\right|_{1,\infty,S} \le K_m N^{6-m} \left\|\xi_x\right\|_{m,S}.$$

Now $\xi_x = \left(Y_m^{p+1}\right)_\eta / J_m^{p+1}$. Moreover by (2.53)

$$\left|M_{i,j}^k\right|_{m,\infty,S} \le C_m \qquad \text{for all } j > N;$$

and by (2.9b)

$$A_1\rho^2 \le \left|J^k_{i,j}\right| \le A_2\rho^2 \qquad \text{for all } j > N.$$

So it is easy to see that

$$\left|\xi_x - \widehat{\xi}_x\right|_{1,\infty,S} \le CN^{-4} \tag{2.57}$$

for all $M^k_{i,j}$ with $j > N$, and N large enough. A similar result holds for ξ_y, η_x and η_y. We are now in a position to prove the following lemma.

Lemma 2.3 *Let γ_s be contained in T^k. Then*

$$\begin{aligned} &2\rho^2\left|\int_{\gamma_s}\left[\frac{\partial u^{p+1}}{\partial y}\frac{d}{ds}\left(\frac{\partial u^{p+1}}{\partial x}\right)\right]ds\right| \\ \le\ & C(\ln N)^2\left(\left\|\left[\left(\frac{\partial u^{p+1}}{\partial x}\right)^a\right]\right\|^2_{1/2,\gamma_s}+\left\|\left[\left(\frac{\partial u^{p+1}}{\partial y}\right)^a\right]\right\|^2_{1/2,\gamma_s}\right) \\ +\ & \frac{\rho^2}{16}\sum_{l=1}^{2}\left(\left|u^{p+1}_m\right|^2_{l,\Omega^{p+1}_m}+\left|u^{p+1}_n\right|^2_{l,\Omega^{p+1}_n}\right). \end{aligned} \tag{2.58a}$$

Here

$$\left\|\left[\left(\frac{\partial u^{p+1}}{\partial x}\right)^a\right]\right\|^2_{1/2,\gamma_s} = \left\|\left(\frac{\partial u^{p+1}_m}{\partial x}\right)^a(1,\eta) - \left(\frac{\partial u^{p+1}_n}{\partial x}\right)^a(0,\eta)\right\|^2_{1/2,(0,1)}, \tag{2.58b}$$

and

$$\left\|\left[\left(\frac{\partial u^{p+1}}{\partial y}\right)^a\right]\right\|^2_{1/2,\gamma_s} = \left\|\left(\frac{\partial u^{p+1}_m}{\partial y}\right)^a(1,\eta) - \left(\frac{\partial u^{p+1}_n}{\partial y}\right)^a(0,\eta)\right\|^2_{1/2,(0,1)}. \tag{2.58c}$$

It is easy to see that

$$\begin{aligned} &\int_{\gamma_s}\frac{\partial u^{p+1}_m}{\partial y}\frac{d}{ds}\left(\frac{\partial u^{p+1}_m}{\partial x}\right)ds - \int_{\gamma_s}\left(\frac{\partial u^{p+1}_m}{\partial y}\right)^a\frac{d}{ds}\left(\frac{\partial u^{p+1}_m}{\partial x}\right)^a ds \\ =\ & \int_0^1\left\{\left(\left(u^{p+1}_m\right)_\xi\left(\xi_y-\widehat{\xi}_y\right)+\left(u^{p+1}_m\right)_\eta\left(\eta_y-\widehat{\eta}_y\right)\right)\frac{d}{d\eta}\right. \\ & \left.\left(\left(u^{p+1}_m\right)_\xi\xi_x+\left(u^{p+1}_m\right)_\eta\eta_x\right)\right\}(1,\eta)\,d\eta \end{aligned} \tag{2.59}$$

$$+ \int_0^1 \left\{ \left(\left(u_m^{p+1}\right)_\xi \left(\widehat{\xi}_y\right) + \left(u_m^{p+1}\right)_\eta \left(\widehat{\eta}_y\right) \right) \frac{d}{d\eta} \left(\left(u_m^{p+1}\right)_\xi \left(\xi_x - \widehat{\xi}_x\right) + \left(u_m^{p+1}\right)_\eta \left(\eta_x - \widehat{\eta}_x\right) \right) \right\} (1,\eta)\, d\eta.$$

Hence

$$2\rho^2 \left| \int_{\gamma_s} \frac{\partial u_m^{p+1}}{\partial y} \frac{d}{ds} \left(\frac{\partial u_m^{p+1}}{\partial x} \right) ds - \int_0^1 \left\{ \left(\left(u_m^{p+1}\right)_\xi \widehat{\xi}_y + \left(u_m^{p+1}\right)_\eta \widehat{\eta}_y \right) \frac{d}{d\eta} \left(\left(u_m^{p+1}\right)_\xi \widehat{\xi}_x + \left(u_m^{p+1}\right)_\eta \widehat{\eta}_x \right) \right\} (1,\eta)\, d\eta \right|$$
$$\leq \frac{C}{N^4} \sum_{l=1}^{2} \left| u_m^{p+1} \right|_{l,\partial S}^2 \leq \frac{C}{N^4} \left(\left\| \left(u_m^{p+1}\right)_\xi \right\|_{3/2,S}^2 + \left\| \left(u_m^{p+1}\right)_\eta \right\|_{3/2,S}^2 \right)$$
$$\leq \frac{C}{N^2} \left(\sum_{l=1}^{2} \left| u_m^{p+1} \right|_{l,\Omega_m^{p+1}}^2 \right). \tag{2.60}$$

by the trace theorem and an inequality for fractional Sobolev spaces we obtain below along with (2.43a - 2.43d). The inequality is as follows.

Let

$$w(\xi,\eta) = \sum_{m=0}^{N} \sum_{n=0}^{N} a_{m,n} \xi^m \eta^n$$

defined on S. Then

$$\|w\|_{1/2,S}^2 \leq C \|w\|_{0,S} \|w\|_{1,S} \leq CN^2 \|w\|_{0,S}^2$$

by the interpolation inequality and the inverse inequality for differentiation in [14]. Thus for N large enough

$$2\rho^2 \left| \int_{\gamma_s} \frac{\partial u_m^{p+1}}{\partial y} \frac{d}{ds} \left(\frac{\partial u_m^{p+1}}{\partial x} \right) ds - \int_{\gamma_s} \left(\frac{\partial u_m^{p+1}}{\partial y} \right)^a \frac{d}{ds} \left(\frac{\partial u_m^{p+1}}{\partial x} \right)^a ds \right|$$
$$\leq \frac{\rho^2}{32} \sum_{l=1}^{2} \left| u_m^{p+1} \right|_{l,\Omega_m^{p+1}}^2. \tag{2.61}$$

Now

$$\left| \int_0^1 \left\{ \left(\frac{\partial u_m^{p+1}}{\partial y} \right)^a \frac{d}{d\eta} \left(\frac{\partial u_m^{p+1}}{\partial x} \right)^a \right\} (1,\eta)\, d\eta - \int_0^1 \left\{ \left(\frac{\partial u_n^{p+1}}{\partial y} \right)^a \frac{d}{d\eta} \left(\frac{\partial u_n^{p+1}}{\partial x} \right)^a \right\} (0,\eta)\, d\eta \right|$$

$$\leq \left|\int_0^1 \left\{\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta)\right\} \frac{d}{d\eta}\left(\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)\right) d\eta\right|$$
$$+ \left|\int_0^1 \left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta) \left\{\frac{d}{d\eta}\left(\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a (0,\eta)\right)\right\} d\eta\right| .$$

Clearly $\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)$, $\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a (1,\eta)$, $\left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a (0,\eta)$ and $\left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta)$ are polynomials in η of degree at most $2N$. Hence by Theorem 2.4

$$\begin{aligned}
(2.62) \qquad & 2\rho^2 \left|\int_0^1 \left\{\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a \frac{d}{d\eta}\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a\right\}(1,\eta)\, d\eta\right. \\
& - \left.\int_0^1 \left\{\left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a \frac{d}{d\eta}\left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a\right\}(0,\eta)\, d\eta\right| \\
& \leq \frac{C}{K}(\ln N)^2 \left\{\left\|\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta)\right\|^2_{1/2,(0,1)}\right. \\
& + \left.\left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta) - \left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a (0,\eta)\right\|^2_{1/2,(0,1)}\right\} \\
& + K\left\{\left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)\right\|^2_{1/2,(0,1)} + \left\|\left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a (0,\eta)\right\|^2_{1/2,(0,1)}\right\},
\end{aligned}$$

for any $K > 0$.

Now

$$\begin{aligned}
&\left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)\right\|_{1/2,(0,1)} \\
&\leq C_1\left(\left\|\widehat{\xi}_x\right\|_{1,\infty,(0,1)} \left\|\left(u_m^{p+1}\right)_\xi (1,\eta)\right\|_{1/2,(0,1)} + \left\|\widehat{\eta}_x\right\|_{1,\infty,(0,1)} \left\|\left(u_m^{p+1}\right)_\eta\right\|_{1/2,(0,1)}\right).
\end{aligned}$$

Using the above estimate, the trace theorem and (2.43a - 2.43d) we get

$$(2.63) \qquad \left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a (1,\eta)\right\|^2_{1/2,(0,1)} \leq C\sum_{l=1}^{2} \left|u_m^{p+1}\right|^2_{l,\Omega_m^{p+1}} .$$

Substituting the above estimates into (2.62) and choosing K small enough we can conclude that

$$\begin{aligned}
(2.64) \qquad & 2\rho^2 \left|\int_0^1 \left\{\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a \frac{d}{d\eta}\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a\right\}(1,\eta)\, d\eta\right. \\
& - \left.\int_0^1 \left\{\left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a \frac{d}{d\eta}\left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a\right\}(0,\eta)\, d\eta\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq C(\ln N)^2\left\{\left\|\left(\frac{\partial u_m^{p+1}}{\partial y}\right)^a(1,\eta)-\left(\frac{\partial u_n^{p+1}}{\partial y}\right)^a(0,\eta)\right\|_{1/2,(0,1)}^2\right.\\
&+\left.\left\|\left(\frac{\partial u_m^{p+1}}{\partial x}\right)^a(1,\eta)-\left(\frac{\partial u_n^{p+1}}{\partial x}\right)^a(0,\eta)\right\|_{1/2,(0,1)}^2\right\}\\
&+\frac{\rho^2}{32}\left(\sum_{l=1}^{2}\left|u_m^{p+1}\right|_{l,\Omega_m^{p+1}}^2+\left|u_n^{p+1}\right|_{l,\Omega_n^{p+1}}^2\right).
\end{aligned}
$$

Hence we get the required result.□

Finally

$$
2\rho^2\left|\int_{\partial\Omega_l^{p+1}\bigcap\Gamma_m}\left(u_l^{p+1}\right)_{\nu_m}\left(u_l^{p+1}\right)_{\sigma_m\sigma_m}d\sigma_m\right|=2\rho^2\left|\int\left\{\left(u_l^{p+1}\right)_{\nu_m}\frac{d}{d\eta}\left(\left(u_l^{p+1}\right)_{\sigma_m}\right)\right\}(1,\eta)\,d\eta\right|
$$

or a similar expression.

Now

$$
\begin{aligned}
\left(u_l^{p+1}\right)_{\sigma_m}(1,\eta) &= A(\eta)\left(u_l^{p+1}\right)_\eta(1,\eta), \qquad \text{and}\\
\left(u_l^{p+1}\right)_{\nu_m}(1,\eta) &= B(\eta)\left(u_l^{p+1}\right)_\xi(1,\eta)+C(\eta)\left(u_l^{p+1}\right)_\eta(1,\eta).
\end{aligned}
$$

The form of the expressions $B(\eta)$ and $C(\eta)$ do not matter except that they are analytic functions of η involving X_l^{p+1}, Y_l^{p+1} and their derivatives at $(1,\eta)$. Hence we can bound the derivatives of B and C as in (2.54a - 2.57). Let $\widehat{A}(\eta)$ be the unique polynomial that is the orthogonal projection of $A(\eta)$ into the space of polynomials of degree $N-1$ with respect to the usual norm defined on $H^2(0,1)$. We now define

$$
\begin{aligned}
\left(u_l^{p+1}\right)_{\sigma_m}^a(1,\eta) &= \widehat{A}(\eta)\left(u_l^{p+1}\right)_\eta(1,\eta), \qquad \text{and}\\
\left(u_l^{p+1}\right)_{\nu_m}^a(1,\eta) &= \widehat{B}(\eta)\left(u_l^{p+1}\right)_\xi(1,\eta)+\widehat{C}(\eta)\left(u_l^{p+1}\right)_\eta(1,\eta).
\end{aligned}
$$

It is easy to prove as we did the estimate (2.58a - 2.58c) that

$$
\text{(2.65a)}\qquad
\begin{aligned}
&2\rho^2\left|\int_{\gamma_s}\left\{\left(u_l^{p+1}\right)_{\nu_m}\left(u_l^{p+1}\right)_{\sigma_m\sigma_m}\right\}d\sigma_m\right|\\
&\leq C(\ln N)^2\left\|\left(u_l^{p+1}\right)_{\sigma_m}^a\right\|_{1/2,\gamma_s}^2+\frac{\rho^2}{32}\left(\sum_{i=1}^{2}\left|u_l^{p+1}\right|_{i,\Omega_l^{p+1}}^2\right),
\end{aligned}
$$

where $\gamma_s \subseteq \Gamma_m \bigcap \partial\Omega_l^{p+1}$ for some $m \in \{k, k+1\}$.

Here

(2.65b)
$$\left\| \left(u_l^{p+1}\right)_{\sigma_m}^a \right\|_{1/2,\gamma_s}^2 = \left\| \left(u_l^{p+1}\right)_{\sigma_m}^a (1,\eta) \right\|_{1/2,(0,1)}^2 .$$

Hence using the estimates (2.58a - 2.58c) and (2.65a - 2.65b) we obtain

(2.66a)
$$|(VI)| \leq (XII) + \sum_{l \in L^k} \frac{\rho^2}{8} \sum_{i=1}^{2} \left| u_l^{p+1}(x,y) \right|_{i,\Omega_l^{p+1}}^2 .$$

where

$$\begin{aligned} (XII) &= C(\ln N)^2 \Bigg\{ \sum_{\gamma_s \subseteq T^k} \left(\left\| \left[\left(u^{p+1}\right)_x^a \right] \right\|_{1/2,\gamma_s}^2 + \left\| \left[\left(u^{p+1}\right)_y^a \right] \right\|_{1/2,\gamma_s}^2 \right) \\ &+ \left(\sum_{m=k}^{k+1} \sum_{l \in L^k} \sum_{\gamma_s \subseteq \partial\Omega_l^{p+1} \bigcap \Gamma_m} \left\| \left(u_l^{p+1}\right)_{\sigma_m}^a \right\|_{\gamma_s}^2 \right) \Bigg\} \\ &- 2\rho^2 \sum_{l \in L^k} \sum_{\gamma_s \subseteq \partial S_c^k \bigcap \partial\Omega_l^{p+1}} \int_{\gamma_s} \frac{\partial u_l^{p+1}}{\partial y} \frac{d}{ds} \left(\frac{\partial u_l^{p+1}}{\partial x} \right) ds. \end{aligned} \tag{2.66b}$$

Now using Lemma 2.4 and (2.9a - 2.9b) we get

(2.67)
$$\begin{aligned} &\sum_{l \in L^k} \left| u_l^{p+1}(x,y) \right|_{0,\Omega_l^{p+1}}^2 \\ &\leq T \Bigg\{ \sum_{\gamma_s \subseteq T^k} \left| \left[u^{p+1} \right] \right|_{0,\gamma_s}^2 + \left(\sum_{m=k}^{k+1} \sum_{l \in L^k} \sum_{\gamma_s \subseteq \partial\Omega_l^{p+1} \bigcap \Gamma_m} \left| u_l^{p+1} \right|_{0,\gamma_s}^2 \right) \\ &+ \sum_{l \in L^k} \left| u_l^{p+1}(x,y) \right|_{1,\Omega_l^{p+1}}^2 \Bigg\} . \end{aligned}$$

Here the constant T is independent of N.

Choose

$$R > \frac{\rho}{\nu - \rho} + \alpha\rho + (T+1)\rho^2 .$$

Adding

$$T\rho^2\left\{\sum_{\gamma_s\subseteq T^k}\left|[u^{p+1}]\right|^2_{0,\gamma_s}+\left(\sum_{m=k}^{k+1}\sum_{l\in L^k}\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\Gamma_m}\left|u_l^{p+1}\right|^2_{0,\gamma_s}\right)\right\}$$

to both sides of (2.50) we obtain

$$\begin{aligned}(2.68)\qquad &\sum_{l\in L^k}\left(\rho^2\left|u_l^{p+1}(x,y)\right|^2_{0,\Omega_l^{p+1}}\right.\\ &+\left.\left(R-\frac{\rho}{\nu-\rho}-\alpha\rho-T\rho^2\right)\left|u_l^{p+1}(x,y)\right|^2_{1,\Omega_l^{p+1}}+\frac{21}{32}\rho^2\left|u_l^{p+1}(x,y)\right|^2_{2,\Omega_l^{p+1}}\right)\\ &\le ((I)+(II)+(III)+(V))+((XI)+(XII))\\ &+T\rho^2\left\{\sum_{\gamma_s\subseteq T^k}\left\|[u^{p+1}]\right\|^2_{0,\gamma_s}+\left(\sum_{m=k}^{k+1}\sum_{l\in L^k}\sum_{\gamma_s\subseteq\partial\Omega_l^{p+1}\bigcap\Gamma_m}\left\|u_l^{p+1}\right\|^2_{0,\gamma_s}\right)\right\}.\end{aligned}$$

We now have to approximate

$$\left|\Delta u_l^{p+1}\right|^2_{0,\Omega_l^{p+1}}=\int_{\Omega_l^{p+1}}\int\left(\Delta u_l^{p+1}\right)^2dxdy.$$

Now

$$\Delta u_l^{p+1}=a_l^{p+1}\left(u_l^{p+1}\right)_{\xi\xi}+2b_l^{p+1}\left(u_l^{p+1}\right)_{\xi\eta}+c_l^{p+1}\left(u_l^{p+1}\right)_{\eta\eta}+d_l^{p+1}\left(u_l^{p+1}\right)_{\xi}+e_l^{p+1}\left(u_l^{p+1}\right)_{\eta}.$$

Hence

$$\int_{\Omega_l^{p+1}}\int\left(\Delta u_l^{p+1}\right)^2dxdy=\int_{(0,1)\times(0,1)}\int\left(L_l^{p+1}u_l^{p+1}\right)^2d\xi d\eta,$$

where

$$L_l^{p+1}w=A_l^{p+1}w_{\xi\xi}+2B_l^{p+1}w_{\xi\eta}+C_l^{p+1}w_{\eta\eta}+D_l^{p+1}w_\xi+E_l^{p+1}w_\eta,$$

and $A_l^{p+1}=a_l^{p+1}\sqrt{J_l^{p+1}}$, etc. Let $\widehat{A}_l^{p+1}$ denote the unique polynomial which is the orthogonal projection of A_l^{p+1} into the space of polynomials of degree $N-1$ in ξ and η with respect to the usual inner product in $H^2(S)$. We define $\widehat{B}_l^{p+1},\widehat{C}_l^{p+1},\widehat{D}_l^{p+1}$ and

$\widehat{E}_l^{p+1}$ in the same way.

Let

$$\left(L_l^{p+1}\right)^a w = \widehat{A}^{p+1} w_{\xi\xi} + 2\widehat{B}_l^{p+1} w_{\xi\eta} + \ldots$$

Then it is easy to prove that for N large enough

$$(2.69) \quad (I) = \sum_{l \in L^k} \left(\rho^2 + \frac{R}{2K}\right) \int_{\Omega_l^{p+1}} \int \left(\Delta u_l^{p+1}\right)^2 dx dy$$
$$\leq \sum_{l \in L^k} 2 \left(\rho^2 + \frac{R}{2K}\right) \int_S \int \left(\left(L_l^{p+1}\right)^a u_l^{p+1}\right)^2 d\xi d\eta + \frac{\rho^2}{16} \left\|u_l^{p+1}\right\|_{2,S}^2 .$$

Substituting $K = \rho^2/2R$ in (II) and estimating the term (V) as before along with the above estimates we obtain

$$(2.70) \quad \sum_{l \in L^k} \frac{\rho^2}{2} \left\|u_l^{p+1}\right\|_{2,S}^2$$
$$\leq C (\ln N)^2 \left(\left(\sum_{l \in L^k} \left\| \left(L_l^{p+1}\right)^a u_l^{p+1} (\xi, \eta) \right\|_{0,S}^2 \right) \right.$$
$$+ \sum_{\gamma_s \subseteq T^k} \left(\left\| \left[u^{p+1}\right] \right\|_{0,\gamma_s}^2 + \left\| \left[\left(u^{p+1}\right)_x^a\right] \right\|_{1/2,\gamma_s}^2 + \left\| \left[\left(u^{p+1}\right)_y^a\right] \right\|_{1/2,\gamma_s}^2 \right)$$
$$\left. + \sum_{m=k}^{k+1} \sum_{\gamma_s \subseteq \partial T^k \bigcap \Gamma_m} \left(\left\|u^{p+1}\right\|_{0,\gamma_s}^2 + \left\| \left(u^{p+1}\right)_{\sigma_m}^a \right\|_{1/2,\gamma_s}^2 \right) \right)$$
$$+ \sum_{\gamma_s \subseteq \partial S_c^k} \left(\int_{\gamma_s} R u^{p+1} \frac{\partial u^{p+1}}{\partial n} ds - 2\rho^2 \int_{\gamma_s} \frac{\partial u^{p+1}}{\partial y} \frac{d}{ds} \left(\frac{\partial u^{p+1}}{\partial x} \right) ds \right) + (III) + (XI) .$$

Here (III) is as defined in (2.42) and (XI) is as defined in (2.45).

We are now in a position to prove an energy inequality for the subdomain S^k which we state in the following theorem.

Theorem 2.2 *Consider the subdomain S^k. Then for N large enough*

$$(2.71) \quad \frac{1}{8} \sum_{j=1}^{N} \sum_{i=1}^{I_k} \left\|u_{i,j}^k (\tau_k, \theta_k)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2 + \frac{\alpha}{8} \sum_{l \in L^k} \left\|u_l^{p+1} (\xi, \eta)\right\|_{2,\Omega_l^{p+1}}^2$$
$$\leq \{(1) + (2) + (3) + (4)\} ,$$

where

$$
\begin{aligned}
(1) &= C(\ln N)^2 \left(\sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| \Delta u_{i,j}^k (\tau_k, \theta_k) \right\|_{0,\widetilde{\Omega}_{i,j}^k}^2 \right. \\
&+ \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Omega}^k} \left(\left\| \left[u^k \right] \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| \left[u_{\tau_k}^k \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| \left[u_{\theta_k}^k \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \\
&+ \left. \sum_{m=k}^{k+1} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Gamma}_m \bigcap \partial \widetilde{\Omega}^k} \left(\left\| u^k \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| u_{\tau_k}^k \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \right), \\
(2) &= C(\ln N)^2 \left(\sum_{\widetilde{\gamma}_l \subseteq \widetilde{B}_\rho^k} \left(\left\| \left[u^k \right] \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| \left[u_{\tau_k}^k \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| \left[u_{\theta_k}^k \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \right), \\
(3) &= C(\ln N)^2 \left(\sum_{l \in L^k} \left\| \left(L_l^{p+1} \right)^a u_l^{p+1} (\xi, \eta) \right\|_{0,S}^2 \right. \\
&+ \sum_{\gamma_s \subseteq T^k} \left(\left\| \left[u^{p+1} \right] \right\|_{0,\gamma_s}^2 + \left\| \left[\left(u^{p+1} \right)_x^a \right] \right\|_{1/2,\gamma_s}^2 + \left\| \left[\left(u^{p+1} \right)_y^a \right] \right\|_{1/2,\gamma_s}^2 \right) \\
&+ \left. \sum_{m=k}^{k+1} \sum_{\gamma_s \subseteq \partial T^k \bigcap \Gamma_m} \left(\left\| u^{p+1} \right\|_{0,\gamma_s}^2 + \left\| \left(u^{p+1} \right)_{\sigma_m}^a \right\|_{1/2,\gamma_s}^2 \right) \right), \text{ and} \\
(4) &= \sum_{m=k}^{k+1} (-1)^{m-k+1} \left(\rho^2 B \left(\psi_1^m, u_{\nu_m}^k, u_{\sigma_m}^k \right) \right) (P_m) \\
&+ \sum_{\gamma_s \subseteq \partial S_c^k} \int_{\gamma_s} \left(R u^k \frac{\partial u^k}{\partial n} - 2\rho^2 \frac{\partial u^k}{\partial y} \frac{d}{ds} \left(\frac{\partial u^k}{\partial x} \right) \right) ds.
\end{aligned}
$$

By (2.43a - 2.43d) there exists a positive constant α such that

$$
\alpha \left\| u_l^{p+1} (\xi, \eta) \right\|_{2,S}^2 \leq \rho^2 \left\| u_l^{p+1} (x, y) \right\|_{2,\Omega_l^{p+1}}^2 \tag{2.72}
$$

for all $\Omega_l^{p+1} \subseteq T^k$ and for all k.

Then combining (2.33) and (2.70) and using (2.72) we obtain

$$
\begin{aligned}
&\frac{1}{2} \sum_{j=1}^{N} \sum_{i=1}^{I_k} \left\| u_{i,j}^k (\tau_k, \theta_k) \right\|_{2,\widetilde{\Omega}_{i,j}^k}^2 + \frac{\alpha}{2} \sum_{l \in L^k} \left\| u_l^{p+1} (\xi, \eta) \right\|_{2,S}^2 \\
&\leq \{(1) + (3) + (4)\} + (XIII) + (XIV) + (XV).
\end{aligned} \tag{2.73}
$$

Here

$$
\begin{aligned}
(XIII) \;=\; & C\ln N\left(\sum_{i=2}^{I_k}\left(\left|\left[\left(u^k\right)_{\tau_k}\right]\left(G_i^k\right)\right|^2+\left|\left[\left(u^k\right)_{\theta_k}\right]\left(G_i^k\right)\right|^2\right)\right. \\
& \left.+\sum_{m=k}^{k+1}\sum_{i=2-\delta_{m,k+1}}^{M_m-\delta_{m,k+1}}\left(\left|\left[\left(u^{p+1}\right)_x\right]\left(D_i^m\right)\right|^2+\left|\left[\left(u^{p+1}\right)_y\right]\left(D_i^m\right)\right|^2\right)\right)
\end{aligned}
$$

The remaining two terms are

$$
\begin{aligned}
(XIV) &= R\left(\sum_{i=1}^{I_k}\int_{\psi_i^k}^{\psi_{i+1}^k}\left(\left(u_{i,N}^k\right)\left(u_{i,N}^k\right)_{\tau_k}-\left(u_{i,N+1}^k\right)\left(u_{i,N+1}^k\right)_{\tau_k}\right)(\ln\rho,\theta_k)\,d\theta_k\right), \text{ and} \\
(XV) &= 2\left(\sum_{i=1}^{I_k}\int_{\psi_i^k}^{\psi_{i+1}^k}\left(\left(u_{i,N}^k\right)_{\theta_k}\left(u_{i,N}^k\right)_{\tau_k\theta_k}-\left(u_{i,N+1}^k\right)_{\theta_k}\left(u_{i,N+1}^k\right)_{\tau_k\theta_k}\right)(\ln\rho,\theta_k)\,d\theta_k\right).
\end{aligned}
$$

Once more using Theorem 2.4 we can show that

$$
\begin{aligned}
(2.74)\qquad & |(XIV)|+|(XV)| \\
\leq\; & C(\ln N)^2\left(\sum_{\tilde{\gamma}_l\subseteq\tilde{B}_\rho^k}\left(\left\|\left[\left(u^k\right)\right]\right\|_{0,\tilde{\gamma}_l}^2+\left\|\left[\left(u^k\right)_{\tau_k}\right]\right\|_{1/2,\tilde{\gamma}_l}^2+\left\|\left[\left(u^k\right)_{\theta_k}\right]\right\|_{1/2,\tilde{\gamma}_l}^2\right)\right) \\
& +\frac{1}{32}\sum_{i=1}^{I_k}\left\|u_{i,N}^k\left(\tau_k,\theta_k\right)\right\|_{2,\tilde{\Omega}_{i,N}^k}^2+\frac{\alpha}{32}\sum_{i=1}^{I_k}\left\|u_{i,N+1}^k(x,y)\right\|_{2,\Omega_{i,N+1}^k}^2.
\end{aligned}
$$

We now consider the term $\left|\left[\left(u^{p+1}\right)_x\right]\left(D_i^m\right)\right|^2$. There exist two domains Ω_t^{p+1} and Ω_s^{p+1} such that $\partial\Omega_s^{p+1}\bigcap\partial\Omega_t^{p+1}=\gamma_l$ and D_i^m is an end point of the curve γ_l. Let us assume that $\gamma_l\bigcap\partial\Omega_t^{p+1}$ is the image of the mapping M_t^{p+1} of the boundary $\eta=1$ of S and $\gamma_l\bigcap\partial\Omega_s^{p+1}$ is the image of the mapping M_s^{p+1} of the boundary $\eta=0$ of S. Further let D_i^m correspond to the image of the point $\xi=0$ for both these cases.

Now

$$
\begin{aligned}
(2.75)\qquad & \left|\left[\left(u^{p+1}\right)_x\right]\left(D_i^m\right)\right|^2 \\
\leq\; & 3\left\{\left(\left(\left(u_t^{p+1}\right)_x^a(\xi,1)-\left(u_s^{p+1}\right)_x^a(\xi,0)\right)\Big|_{\xi=0}\right)^2\right. \\
& \left.+\left(\left(\left(u_t^{p+1}\right)_x-\left(u_t^{p+1}\right)_x^a\right)(\xi,1)\Big|_{\xi=0}\right)^2+\left(\left(\left(u_s^{p+1}\right)_x-\left(u_s^{p+1}\right)_x^a\right)(\xi,0)\Big|_{\xi=0}\right)^2\right\}.
\end{aligned}
$$

Moreover $\left(\left(u_t^{p+1}\right)_x^a(\xi,1)-\left(u_s^{p+1}\right)_x^a(\xi,0)\right)$ is a polynomial in ξ of degree at most $2N$. Now by Theorem 4.79 of [42] we have that if $p(s)$ is polynomial of degree M defined on $[0,1]$ then

$$\|p\|^2_{L^\infty[0,1]} \le C(1+\ln M)\,\|p\|^2_{1/2,[0,1]}\,.$$

Hence we obtain

$$3\left(\left(\left(u_t^{p+1}\right)_x^a(\xi,1)-\left(u_s^{p+1}\right)_x^a(\xi,0)\right)\Big|_{\xi=0}\right)^2 \le K\ln N\left\|\left[\left(u^{p+1}\right)_x^a\right]\right\|^2_{1/2,\gamma_l}.$$

Now

$$\begin{aligned}
&\left|\left(\left(u_t^{p+1}\right)_x-\left(u_t^{p+1}\right)_x^a\right)(0,1)\right|^2\\
\le\ & 2\left(\left|\left(\left(u_t^{p+1}\right)_\xi\left(\xi_x-\widehat{\xi}_x\right)\right)(0,1)\right|^2+\left|\left(\left(u_t^{p+1}\right)_\eta\left(\eta_x-\widehat{\eta}_x\right)\right)(0,1)\right|^2\right).
\end{aligned}$$

Using (2.57) and the Sobolev's embedding theorem we can conclude that

$$\left|\left(\left(u_t^{p+1}\right)_x-\left(u_t^{p+1}\right)_x^a\right)(0,1)\right|^2 \le \frac{C}{N^4}\left\|u_t^{p+1}(\xi,\eta)\right\|^2_{5/2,S}.$$

And as before we can show

$$\left|\left(\left(u_t^{p+1}\right)_x-\left(u_t^{p+1}\right)_x^a\right)(0,1)\right|^2 \le \frac{C}{N^2}\left\|u_t^{p+1}(\xi,\eta)\right\|^2_{2,S}.$$

Choosing N large enough we obtain

$$\left|\left(\left(u_t^{p+1}\right)_x-\left(u_t^{p+1}\right)_x^a\right)(0,1)\right|^2 \le \frac{\alpha}{32}\left\|u_t^{p+1}(\xi,\eta)\right\|^2_{2,S}.$$

And so we can conclude that

$$\begin{aligned}
(2.76)\qquad & C\ln N\left(\sum_{m=k}^{k+1}\left(\left|\left[\left(u^{p+1}\right)_x\right](D_i^m)\right|^2+\left|\left[\left(u^{p+1}\right)_y\right](D_i^m)\right|^2\right)\right)\\
\le\ & K(\ln N)^2\left(\sum_{m=k}^{k+1}\sum_{\gamma_s\subseteq\partial T^k\bigcap\Gamma_m}\left(\left\|\left[\left(u^{p+1}\right)_x^a\right]\right\|^2_{1/2,\gamma_s}+\left\|\left[\left(u^{p+1}\right)_y^a\right]\right\|^2_{1/2,\gamma_s}\right)\right)\\
+\ & \frac{12\alpha}{32}\sum_{l\in L^k}\left\|u_l^{p+1}\right\|^2_{2,S}.
\end{aligned}$$

In the same way, we can conclude that

$$C \ln N \sum_{i=2}^{I_k} \left(\left| \left[\left(u^k\right)_{\tau_k} \right] \left(G_i^k\right) \right|^2 + \left| \left[\left(u^k\right)_{\theta_k} \right] \left(G_i^k\right) \right|^2 \right) \tag{2.77}$$
$$\leq K (\ln N)^2 \left(\sum_{\widetilde{\gamma}_l \subseteq \widetilde{B}_p^k} \left(\left\| \left[\left(u^k\right)_{\tau_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| \left[\left(u^k\right)_{\theta_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \right).$$

Substituting (2.74), (2.76) and (2.77) into (2.73) we get the required result.□

We have now obtained an energy inequality for any of the p subdomains S^k into which we had divided our original domain. Combining these estimates we can now prove the main theorem of this chapter which can be interpreted as a stability estimate for the whole domain.

Theorem 2.3 *Consider the whole domain Ω. Then for N large enough there exists a constant C such that*

$$\sum_{k=1}^{p} \sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| u_{i,j}^k (\tau_k, \theta_k) \right\|_{2,\widetilde{\Omega}_{i,j}^k}^2 + \sum_{l=1}^{L} \left\| u_l^{p+1} (\xi, \eta) \right\|_{2,S}^2 \tag{2.78}$$
$$\leq C (\ln N)^2 \left\{ \left(\sum_{k=1}^{p} \sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| \Delta u_{i,j}^k (\tau_k, \theta_k) \right\|_{0,\widetilde{\Omega}_{i,j}^k}^2 \right.\right.$$
$$+ \sum_{k=1}^{p} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Omega}^k} \left(\left\| \left[\left(u^k\right) \right] \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| \left[\left(u^k\right)_{\tau_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| \left[\left(u^k\right)_{\theta_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right)$$
$$\left. + \sum_{k=1}^{p} \sum_{m=k}^{k+1} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Gamma}_m \bigcap \partial\widetilde{\Omega}^k} \left(\left\| \left(u^k\right) \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| \left(u^k\right)_{\tau_k} \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \right)$$
$$+ \left(\sum_{k=1}^{p} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{B}_p^k} \left(\left\| \left[\left(u^k\right) \right] \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| \left[\left(u^k\right)_{\tau_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| \left[\left(u^k\right)_{\theta_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \right)$$
$$+ \left(\sum_{l=1}^{L} \left\| \left(\left(L_l^{p+1}\right)^a u_l^{p+1} \right) (\xi, \eta) \right\|_{0,S}^2 \right.$$
$$+ \sum_{\gamma_s \subseteq \Omega^{p+1}} \left(\left\| \left[\left(u^{p+1}\right) \right] \right\|_{0,\gamma_s}^2 + \left\| \left[\left(u^{p+1}\right)_x^a \right] \right\|_{1/2,\gamma_s}^2 + \left\| \left[\left(u^{p+1}\right)_y^a \right] \right\|_{1/2,\gamma_s}^2 \right)$$
$$\left.\left. + \sum_{k=1}^{p} \sum_{\gamma_s \subseteq \partial\Omega^{p+1} \bigcap \Gamma_k} \left(\left\| \left(u^{p+1}\right) \right\|_{0,\gamma_s}^2 + \left\| \left(u^{p+1}\right)_{\sigma_k}^a \right\|_{1/2,\gamma_s}^2 \right) \right) \right\}.$$

Summing the estimate (2.71) in Theorem (2.2) over k and estimating terms as before the result follows.□

2.4 Technical Results

In this section we prove the results which we frequently refer to in Section 2.3.

Lemma 2.4 *Let $w(\theta)$ be a piecewise smooth function defined for $\theta \in [\theta_1, \ldots \theta_{M+1}]$ which has discontinuities only at the points $\theta_2, \theta_3, \ldots \theta_M$. Then*

$$\begin{aligned}\int_{\theta_1}^{\theta_{M+1}} w^2(\theta)\, d\theta \;&\leq\; 2\left(\frac{(\theta_{M+1}-\theta_1)^2}{2}\sum_{k=1}^{M}\int_{\theta_k}^{\theta_{k+1}}\left(\frac{dw}{d\theta}\right)^2 d\theta\right.\\ &+\left. M(\theta_{M+1}-\theta_1)\left(w^2(\theta_1)+\sum_{j=2}^{M}\left(w\left(\theta_j^+\right)-w\left(\theta_j^-\right)\right)^2\right)\right). \qquad (2.79)\end{aligned}$$

Here

$$\begin{aligned} w\left(\theta_j^+\right) &= \lim_{\theta>\theta_j, \theta\to\theta_j} w(\theta), \quad \text{and}\\ w\left(\theta_j^-\right) &= \lim_{\theta<\theta_j, \theta\to\theta_j} w(\theta).\end{aligned}$$

Define a function $s(\theta)$ as follows:

$$s(\theta) = \begin{cases} w(\theta_1) & \text{for } \theta_1 \leq \theta < \theta_2,\\ w(\theta_1) + \sum_{j=2}^{k}\left(w\left(\theta_j^+\right) - w\left(\theta_j^-\right)\right) & \text{for } \theta_k \leq \theta < \theta_{k+1},\ 2 \leq k \leq M\end{cases}.$$

Then $w(\theta)$ may be written as

$$w(\theta) = h(\theta) + s(\theta)$$

where $h(\theta)$ is a continuous function which is differentiable a.e.

Moreover $h(\theta_1) = 0$. Now

$$\int_{\theta_1}^{\theta_{M+1}} w^2(\theta)\, d\theta \leq 2\left(\int_{\theta_1}^{\theta_{M+1}} h^2(\theta)\, d\theta + \int^{\theta_{M+1}} s^2(\theta)\, d\theta\right)$$

Clearly

$$h(\theta) = \int_{\theta_1}^{\theta} \frac{dh}{d\phi} d\phi.$$

Hence

$$h^2(\theta) \leq (\theta - \theta_1) \int_{\theta_1}^{\theta_{M+1}} \left(\frac{dh}{d\theta}\right)^2 d\theta.$$

From which we can conclude that

$$\int_{\theta_1}^{\theta_{M+1}} h^2(\theta)\, d\theta \leq \frac{(\theta_{M+1} - \theta_1)^2}{2} \int_{\theta_1}^{\theta_{M+1}} \left(\frac{dh}{d\theta}\right)^2 d\theta.$$

Now

$$\begin{aligned} \int_{\theta_1}^{\theta_{M+1}} s^2(\theta)\, d\theta &\leq (w(\theta_1))^2 \Delta\theta_1 + \sum_{k=2}^{M} k\Delta\theta_k \left((w(\theta_1))^2 + \sum_{j=2}^{k} \left(w\left(\theta_j^+\right) - w\left(\theta_j^-\right)\right)^2 \right) \\ &\leq M(\theta_{M+1} - \theta_1) \left((w(\theta_1))^2 + \sum_{j=2}^{M} \left(w\left(\theta_j^+\right) - w\left(\theta_j^-\right)\right)^2 \right). \end{aligned}$$

And so we obtain the estimate.□

Theorem 2.4 *Let $a^p(s)$ and $b^p(s)$ be polynomials of degree P on the finite interval $[\alpha, \beta]$. Then*

$$\left| \int_{\alpha}^{\beta} a^P(s) \frac{db^P(s)}{ds} ds \right| \leq C \ln P \left\|a^P\right\|_{\frac{1}{2},(\alpha,\beta)} \left\|b^P\right\|_{\frac{1}{2},(\alpha,\beta)}. \tag{2.80}$$

Here $\|.\|_{s,\Omega}$ denotes the *fractional Sobolev norm* on $H^s(\Omega)$ as defined in [23], when s is not an integer.

Now for any $0 < \epsilon < \frac{1}{2}$ we have

$$\left| \int_{\alpha}^{\beta} a^P(s) \frac{db^P(s)}{ds} ds \right| \leq \left\|a^P\right\|_{\frac{1}{2}-\epsilon,(\alpha,\beta)} \left\|\frac{db^P}{ds}\right\|_{-\frac{1}{2}+\epsilon,(\alpha,\beta)} \tag{2.81}$$

since the space of infinitely differentiable functions with compact support in (α, β) is dense in $W_q^t(\alpha, \beta)$ for $0 \leq t \leq \frac{1}{q}$ by Theorem 1.4.2.4 of [23].

Next using Theorem 1.4.4.6 of [23] we have the result that the differentiation operator is a continuous linear operator from $W_q^t(\alpha,\beta)$ to $W_q^{t-1}(\alpha,\beta)$, except when $t=\frac{1}{q}$, with norm proportional to $\frac{1}{\left|t-\frac{1}{q}\right|}$. Thus we can conclude that

$$\left\|\frac{db^P}{ds}\right\|_{-\frac{1}{2}+\epsilon,(\alpha,\beta)} \leq \frac{K}{\epsilon}\left\|b^P\right\|_{\frac{1}{2}+\epsilon,(\alpha,\beta)}. \tag{2.82}$$

Now by the interpolation inequality from [23]

$$\left\|b^P\right\|_{\frac{1}{2}+\epsilon,(\alpha,\beta)} \leq C\left\|b^P\right\|_{\frac{1}{2},(\alpha,\beta)}^{1-2\epsilon}\left\|b^P\right\|_{1,(\alpha,\beta)}^{2\epsilon}. \tag{2.83}$$

And by the inverse inequality for differentiation in [14]

$$\left\|b^P\right\|_{2,(\alpha,\beta)} \leq CP^2\left\|b^P\right\|_{1,(\alpha,\beta)}. \tag{2.84}$$

Once more by the interpolation inequality

$$\left\|b^P\right\|_{1,(\alpha,\beta)} \leq C\left\|b^P\right\|_{\frac{1}{2},(\alpha,\beta)}^{1-\frac{1}{3}}\left\|b^P\right\|_{2,(\alpha,\beta)}^{\frac{1}{3}};$$

and from (2.84) we can conclude that

$$\left\|b^P\right\|_{1,(\alpha,\beta)} \leq CP^{2/3}\left\|b^P\right\|_{1,(\alpha,\beta)}^{\frac{1}{3}}\left\|b^P\right\|_{\frac{1}{2},(\alpha,\beta)}^{1-\frac{1}{3}}.$$

This gives us the inverse inequality for fractional Sobolev norms

$$\left\|b^P\right\|_{1,(\alpha,\beta)} \leq CP\left\|b^P\right\|_{\frac{1}{2},(\alpha,\beta)}. \tag{2.85}$$

Using (2.83) and (2.85) we get

$$\left\|b^P\right\|_{\frac{1}{2}+\epsilon,(\alpha,\beta)} \leq CP^{2\epsilon}\left\|b^P\right\|_{\frac{1}{2},(\alpha,\beta)}. \tag{2.86}$$

Next it is easy to see that

$$\left\|a^P\right\|_{\frac{1}{2}-\epsilon,(\alpha,\beta)} \leq C \left\|a^P\right\|_{\frac{1}{2},(\alpha,\beta)}. \tag{2.87}$$

Substituting the relations (2.82), (2.86) and (2.87) in (2.81) we get

$$\left|\int_\alpha^\beta a^P(s) \frac{db^P(s)}{ds} ds\right| \leq \frac{K}{\varepsilon} P^{2\varepsilon} \left\|a^P\right\|_{\frac{1}{2},(\alpha,\beta)} \left\|b^P\right\|_{\frac{1}{2},(\alpha,\beta)}. \tag{2.88}$$

Taking the minimum over positive ε we get the required result.□

Chapter 3

Computational Techniques for Dirichlet Problems

3.1 Introduction

In Chapter 2 we have introduced a set of local coordinate systems, which we shall refer to as *modified polar coordinates*, in a sectoral neighborhood of every corner and a global coordinate system elsewhere. These modified polar coordinate systems were first introduced by Kondratiev in [30]. We have also derived *differentiability estimates* for the solution of the Dirichlet problem in a polygonal domain in terms of these new systems of coordinates. We also proved a *stability theorem* for a *non-conforming* spectral element representation of the solution which in a sense replicates the *a'priori estimates* we obtain for Elliptic Boundary Value Problems.

We now seek a solution to the Dirichlet problem as in [17] which minimizes a weighted L^2 norm of the residuals in the partial differential equation and a *fractional Sobolev norm* of the residuals in the boundary conditions and enforce continuity by adding a term which measures the jump in the function and its derivatives at *inter-element* boundaries in an appropriate fractional Sobolev norm to the functional being minimized.

To solve the minimization problem we need to solve the *normal equations* for the *least-squares problem.* To compute the residuals in the normal equations we do not need to compute *mass and stiffness matrices* as we have to in the FEM. Instead we collocate the partial differential equation on a finer grid of *Legendre-Gauss-Lobatto* points; then we apply the adjoint differential operator to these residuals and "*project*" these values back to the original grid. Such a treatment can obviously come only from an integration by parts procedure and hence leads to evaluation of terms at the boundaries. These terms can be evaluated by a collocation procedure and the other boundary terms which measures the jump in the function and its derivatives at inter-element boundaries in an appropriate Sobolev norm and a fractional Sobolev norm of the residuals in the boundary conditions can be easily evaluated.

Moreover we show that we do not need to filter the coefficients of the differential and boundary operators or the data. Of course, the evaluation of these residuals on each processor requires the interchange of boundary values between neighboring processors. Hence the communication involved is quite small. Thus we can compute the residuals in the normal equations cheaply and efficiently.

It has been shown in [18] that the methodology can be used to compute the residual in the p and h-p version of the FEM without having to compute any matrices.

We solve the normal equations by the *preconditioned conjugate gradient method* using a block diagonal preconditioner and this is nearly optimal as the *condition number* of the preconditioned system is *polylogarithmic* in N, the number of processors and the number of degrees of freedom in each element. Finally we show that if our data is analytic then the error we commit is *exponentially small* in N.

The spectral element functions we use are non-conforming and hence there is no set of *common boundary values* to solve. Hence the method is highly efficient for solving problems with Dirichlet boundary conditions.

We now briefly review the contents of this chapter. In Section 3.2 we describe the *symmetric formulation* of the numerical scheme and present in algorithmic form

the steps involved in computing the residuals in the normal equations. In Section 3.3 we describe parallelization techniques and the construction of preconditioners for solving the normal equations by the preconditioned conjugate gradient method using a parallel computer. In Section 3.4 we obtain error estimates and show that the error in the numerical solution is exponentially small in N. Finally in Section 3.5 we provide computational results.

In Chapter 4 we shall show how to solve the Poisson equation on a polygonal domain with mixed Neumann and Dirichlet boundary conditions. Finally in Chapter 5 we generalize all our results and show how to solve an elliptic boundary value problem with analytic coefficients on a curvilinear polygon.

3.2 The Numerical Scheme and Symmetric Formulation

In Chapter 2 we had divided the polygonal domain Ω into p sectors $\Omega^1, \Omega^2, \ldots, \Omega^p$ and a remaining portion Ω^{p+1}. We had further divided each of these subdomains into still smaller elements $\left\{\Omega^k_{i,j} \,|1 \le i \le I_{k,j},\ 1 \le j \le J_k,\ 1 \le k \le p\right\}$. It is important to note here that we divide these elements into $p+1$ subsets $\left\{\Omega^k_{i,j} \,|1 \le i \le I_k,\ 1 \le j \le N\right\}_{1 \le k \le p}$ and $\left\{\Omega^k_{i,j} \,|1 \le i \le I_{k,j},\ N < j \le J_k\right\}_{1 \le k \le p}$. In Chapter 2 we had introduced a spectral element representation on each of these subdomains. To be specific, consider $\left\{\Omega^k_{i,j} \,|1 \le i \le I_k,\ 1 \le j \le N\right\}$. We map this $\Omega^k_{i,j}$ onto the subdomain

$$\widetilde{\Omega}^k_{i,j} = \left\{(\tau_k, \theta_k) \,\middle|\eta^k_j < \tau_k < \eta^k_{j+1},\ \psi^k_i < \theta_k < \psi^k_{i+1}\right\}.$$

We map the remaining domains $\left\{\overline{\Omega}^k_{i,j} \,|1 \le i \le I_{k,j},\ N < j \le J_k,\ 1 \le k \le p\right\}$ onto the unit square $S = \{(\xi, \eta) \,|0 \le \xi \le 1,\ 0 \le \eta \le 1\}$ since there is a bijective mapping $M^k_{i,j}$ from $\overline{S}$ onto $\overline{\Omega}^k_{i,j}$. We now define a non-conforming spectral element representation on each of these subdomains as follows:

Let

$$u_{i,j}^{k}(\tau_k,\theta_k)=0$$

for $j=1$, $1\le i\le I_k$, $1\le k\le p$, and

$$u_{i,j}^{k}(\tau_k,\theta_k)=\sum_{m=1}^{N_j}\sum_{n=1}^{N_j}a_{m,n}\tau_k^m\theta_k^n$$

for $1<j\le N$, $1\le i\le I_{k,j}$, $1\le k\le p$. Here $1<N_j\le N$ and we shall put precise bounds on N_j in Section 3.4.

We shall relabel the remaining subdomains

$$\left\{\Omega_{i,j}^{k}\,|1\le i\le I_{k,j},\ N<j\le J_k,\ 1\le k\le p\right\}$$

as $\left\{\Omega_l^{p+1}\,|1\le l\le L\right\}$. Then we define $u_{i,j}^k$ on the unit square onto which $\Omega_{i,j}^k$ is mapped by $\left(M_{i,j}^k\right)^{-1}$ as

$$u_{i,j}^{k}(\xi,\eta)=\sum_{m=1}^{N}\sum_{n=1}^{N}a_{m,n}\xi^m\eta^n.$$

We are now in a position to describe our numerical scheme, which is based on the stability theorem 2.3.

Let $f_{i,j}^k=f\left(X_{i,j}^k(\xi,\eta),Y_{i,j}^k(\xi,\eta)\right)$, for $1\le k\le p$, $N+1\le j\le J_k$, $1\le i\le I_{k,j}$, where $(\xi,\eta)\in S$. Let $\widetilde{f}_{i,j}^k(\xi,\eta)$ denote the polynomial of degree $2N-1$ in ξ and η which is the orthogonal projection of $f_{i,j}^k(\xi,\eta)$ into the space of polynomials of degree $2N-1$ with respect to the usual inner product on $H^2(S)$.
Next, let the vertex $A_k=(x_k,y_k)$. Let $F_{i,j}^k(\tau_k,\theta_k)=e^{2\tau_k}f\left(x_k+e^{\tau_k}\cos\theta_k,y_k+e^{\tau_k}\sin\theta_k\right)$, for $1\le k\le p$, $1\le j\le N$, $1\le i\le I_k$. Here $\eta_j^k\le\tau_k\le\eta_{j+1}^k$ and $\psi_i^k\le\theta_k\le\psi_{i+1}^k$.
Consider first the case $j\neq 1$. We shall let $\widetilde{F}_{i,j}^k(\tau_k,\theta_k)$ denote the polynomial of degree $2N_j-1$ in τ_k and θ_k which is the orthogonal projection of $F_{i,j}^k(\tau_k,\theta_k)$ into the space of

polynomials of degree $2N_j - 1$ with respect to the usual inner product on $H^2\left(\widetilde{\Omega}^k_{i,j}\right)$.

Consider now the case $j = 1$. Let $\widetilde{F}^k_{i,1}(\tau_k, \theta_k) = 0$.

We now consider the boundary condition $u = g_k$ on Γ_k. Let

$$l^k_{1,j}(\tau_k) = g_k\left(x_k + e^{\tau_k}\cos\psi^k_1, y_k + e^{\tau_k}\sin\psi^k_1\right)$$

for $\eta^k_j \leq \tau_k \leq \eta^k_{j+1}$ and $j > 1$. Let $\widetilde{l}^k_{1,j}(\tau_k)$ be the orthogonal projection of $l^k_{1,j}(\tau_k)$ into the space of polynomials of degree $2N_j - 1$ with respect to the usual inner product on $H^2\left(\eta^k_j, \eta^k_{j+1}\right)$ for $j > 1$. Let $\widetilde{l}^k_{1,1}(\tau_k) = 0$.

Let $\Gamma_k \bigcap \partial\Omega^{p+1}_t = C^{p+1}_t$ be the image of the mapping M^{p+1}_t of S onto $\overline{\Omega}^{p+1}_t$ corresponding to $\xi = 0$. Let $l^k_t(\eta) = g_k\left(X^{p+1}_t(0,\eta), Y^{p+1}_t(0,\eta)\right)$, where $0 \leq \eta \leq 1$. We shall let $\widetilde{l}^k_t(\eta)$ denote the polynomial of degree $2N - 1$ which is the orthogonal projection of $l^k_t(\eta)$ with respect to the usual inner product in $H^2(0,1)$.

Finally, we have to consider the boundary condition $u = g_k$ on $\Gamma_k \bigcap \partial\Omega_{k-1}$. Let

$$l^k_{2,j}(\tau_{k-1}) = g_k\left(x_{k-1} + e^{\tau_{k-1}}\cos\left(\psi^{k-1}_{I_{k-1}+1}\right), y_{k-1} + e^{\tau_{k-1}}\sin\left(\psi^{k-1}_{I_{k-1}+1}\right)\right)$$

for $\eta^{k-1}_j \leq \tau_{k-1} \leq \eta^{k-1}_{j+1}$ and $1 \leq j \leq N$. Once more we let $\widetilde{l}^k_{2,j}(\tau_{k-1})$ denote the polynomial of degree $2N_j - 1$ which is the orthogonal projection of $l^k_{2,j}(\tau_{k-1})$ into the space of polynomials of degree $2N_j - 1$ with respect to the usual inner product in $H^2\left(\eta^{k-1}_j, \eta^{k-1}_{j+1}\right)$ for $2 \leq j \leq N$. Let $\widetilde{l}^k_{2,1}(\tau_{k-1}) = 0$.

Our numerical scheme may now be formulated as follows:

Find $\left\{\left\{u^k_{i,j}(\tau_k, \theta_k)\right\}_{1\leq k\leq p, 1\leq i\leq I_k, 1\leq j\leq N}, \left\{u^k_{i,j}(\xi, \eta)\right\}_{1\leq k\leq p, N+1\leq j\leq J_k, 1\leq i\leq I_{k,j}}\right\}$ which minimizes the functional

$$\mathfrak{r}^N\left(\left\{v^k_{i,j}(\tau_k, \theta_k)\right\}_{i,j,k}, \left\{v^k_{i,j}(\xi, \eta)\right\}_{i,j,k}\right) \tag{3.1}$$
$$= \left(\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\Delta v^k_{i,j}(\tau_k, \theta_k) - \widetilde{F}^k_{i,j}(\tau_k, \theta_k)\right\|^2_{0,\widetilde{\Omega}^k_{i,j}}\right.$$

$$+ \sum_{k=1}^{p} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Omega}^k} \left(\left\| [v^k] \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| \left[(v^k)_{\tau_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| \left[(v^k)_{\theta_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right)$$

$$+ \sum_{k=1}^{p} \sum_{m=k}^{k+1} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Gamma}_m \bigcap \partial \widetilde{\Omega}^k} \left(\left\| v^k - \widetilde{l}^m_{m-k+1} \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| (v^k)_{\tau_k} - \left(\widetilde{l}^m_{m-k+1} \right)_{\tau_k} \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \Bigg)$$

$$+ \left(\sum_{k=1}^{p} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{B}^k_\rho} \left(\| [(v)] \|_{0,\widetilde{\gamma}_l}^2 + \left\| \left[(v^k)_{\tau_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| \left[(v^k)_{\theta_k} \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \right)$$

$$+ \left(\sum_{l=1}^{L} \left\| \left(\left(L_l^{p+1} \right)^a v_l^{p+1} \right) (\xi, \eta) - \widetilde{f}_l^{p+1} (\xi, \eta) \right\|_{0,S}^2 \right.$$

$$+ \sum_{\gamma_s \subseteq \Omega^{p+1}} \left(\left\| [v^{p+1}] \right\|_{0,\gamma_s}^2 + \left\| \left[(v^{p+1})_x^a \right] \right\|_{1/2,\gamma_s}^2 + \left\| \left[(v^{p+1})_y^a \right] \right\|_{1/2,\gamma_s}^2 \right)$$

$$+ \sum_{k=1}^{p} \sum_{\gamma_s \subseteq \partial \Omega^{p+1} \bigcap \Gamma_k} \left(\left\| v^{p+1} - \widetilde{l}^k \right\|_{0,\gamma_s}^2 + \left\| (v^{p+1})_{\sigma_k}^a - \left(\widetilde{l}^k \right)_{\sigma_k} \right\|_{1/2,\gamma_s}^2 \right) \Bigg) .$$

This functional we have introduced is closely related to the right hand side of the stability estimate stated in Theorem 2.3. In minimizing the functional

$$\mathfrak{r}^N \left(\left\{ v_{i,j}^k (\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ v_{i,j}^k (\xi, \eta) \right\}_{i,j,k} \right)$$

we seek a solution which minimizes the sum of weighted L^2 norms of the residuals in the partial differential equation and a fractional Sobolev norm of the residuals in the boundary conditions and enforce continuity by adding a term which measures the sum of the squares of the jumps in the function and its derivatives at inter-element boundaries, in an appropriate Sobolev norm.

The above method is essentially a least-squares method and we can obtain a solution by using preconditioned conjugate gradient techniques for solving the normal equations. To be able to do so we must be able to compute the residuals in the normal equations inexpensively. The solution we are seeking minimizes $\mathfrak{r}^N \left(\left\{ v_{i,j}^k (\tau_k, \theta_k) \right\}, \left\{ v_{i,j}^k (\xi, \eta) \right\} \right)$. Now

$$\mathfrak{r}^N (U + \varepsilon V) = \mathfrak{r}^N (U) + 2 \varepsilon V^t (SU - TG) + O \left(\varepsilon^2 \right)$$

for all V, where U is a vector assembled from the values of

$$\left\{\left\{\left\{u_{i,j}^k\left(\tau_{k,j,l}^{N_j},\theta_{k,i,m}^{N_j}\right)\right\}_{0\leq l,m\leq N_j}\right\}_{2\leq j\leq N,1\leq i\leq I_k},\right.$$

$$\left.\left\{\left\{u_{i,j}^k\left(\xi_l^N,\eta_m^N\right)\right\}_{0\leq l,m\leq N}\right\}_{N<j\leq J_k,1\leq i\leq I_{k,j}}\right\}_{1\leq k\leq p}.$$

V is a vector similarly assembled and G is assembled from the data. Here S and T denote matrices. Thus we have to solve $SU - TG = 0$; and so we have to be able to compute $SV - TG$ economically during the iterative process. We now show how this can be done.

We first show how to compute

$$\int_{(\xi_i^k,\xi_{i+1}^k)\times(\eta_j^k,\eta_{j+1}^k)}\int \widehat{L}_{i,j}^k\widehat{v}_{i,j}^k\left(\widehat{L}_{i,j}^k\widehat{u}_{i,j}^k-\widehat{f}_{i,j}^k\right)d\xi d\eta$$

where we may take $1 \leq k \leq p$, $2 \leq j \leq J_k$, $1 \leq i \leq I_{k,j}$. Here

$$\widehat{L}_{i,j}^k v = \widehat{A}_{i,j}^k v_{\xi\xi} + \widehat{B}_{i,j}^k v_{\xi\eta} + \widehat{C}_{i,j}^k v_{\eta\eta} + \widehat{D}_{i,j}^k v_\xi + \widehat{E}_{i,j}^k v_\eta + \widehat{F}_{i,j}^k v$$

where the coefficients $\widehat{A}_{i,j}^k, \ldots, \widehat{E}_{i,j}^k, \widehat{F}_{i,j}^k$ are polynomials of degree $N_j - 1$, $\widehat{u}_{i,j}^k$ and $\widehat{v}_{i,j}^k$ are polynomials of degree N_j and $\widehat{f}_{i,j}^k$ is a polynomial of degree $2N_j - 1$ in each variable.

Define the new variables

$$\begin{aligned} s &= 2\left(\xi-\xi_i^k\right)/\left(\xi_{i+1}^k-\xi_i^k\right)-1\,, \\ t &= 2\left(\eta-\eta_j^k\right)/\left(\eta_{j+1}^k-\eta_j^k\right)-1\,. \end{aligned}$$

Then

$$\begin{aligned} &\int_{(\xi_i^k,\xi_{i+1}^k)\times(\eta_j^k,\eta_{j+1}^k)}\int \widehat{L}_{i,j}^k\widehat{v}_{i,j}^k\left(\widehat{L}_{i,j}^k\widehat{u}_{i,j}^k-\widehat{f}_{i,j}^k\right)d\xi d\eta \\ &= \int_{(-1,1)\times(-1,1)}\int \left(L_{i,j}^k\right)^a v_{i,j}^k\left(\left(L_{i,j}^k\right)^a u_{i,j}^k-\widetilde{g}_{i,j}^k\right)dsdt. \end{aligned}$$

Here

$$\left(L_{i,j}^{k}\right)^{a} w = A_{i,j}^{k} w_{ss} + B_{i,j}^{k} w_{st} + C_{i,j}^{k} w_{tt} + D_{i,j}^{k} w_{s} + E_{i,j}^{k} w_{t} + F_{i,j}^{k} w$$

where the coefficients $A_{i,j}^{k}, \ldots, F_{i,j}^{k}$ are polynomials of degree $N_j - 1$ and $\widetilde{g}_{i,j}^{k}$ is a polynomials of degree $2N_j - 1$.

Moreover

$$\begin{aligned} v_{i,j}^{k}(s,t) &= \widehat{v}_{i,j}^{k}\left(\xi(s,t), \eta(s,t)\right) = \sum_{l=0}^{N_j} \sum_{m=0}^{N_j} a_{m,l} s^{m} t^{l}, \text{ and} \\ u_{i,j}^{k}(s,t) &= \widehat{u}_{i,j}^{k}\left(\xi(s,t), \eta(s,t)\right). \end{aligned}$$

The differential operator $\left(L_{i,j}^{k}\right)^{a}$ is obtained from the differential operator

$$\mathcal{L}_{i,j}^{k} w = \mathcal{A}_{i,j}^{k} w_{ss} + \mathcal{B}_{i,j}^{k} w_{st} + \mathcal{C}_{i,j}^{k} w_{tt} + \mathcal{D}_{i,j}^{k} w_{s} + \mathcal{E}_{i,j}^{k} w_{t} + \mathcal{F}_{i,j}^{k} w$$

with analytic coefficients by choosing $A_{i,j}^{k}$ so that it is the orthogonal projection of $\mathcal{A}_{i,j}^{k}$ into the space of polynomials of degree $N_j - 1$ with respect to the usual inner product in $H^2(-1,1)$. The other coefficients are similarly defined.

In what follows we shall drop the sub and super-scripts and denote $\left(L_{i,j}^{k}\right)^{a}$ by L etc. Thus

$$Lw = Aw_{ss} + Bw_{st} + Cw_{tt} + Dw_{s} + Ew_{t} + Fw.$$

Let L^t denote the formal adjoint of the differential operator L. Then

$$L^t w = (Aw)_{ss} + (Bw)_{st} + (Cw)_{tt} - (Dw)_{s} - (Ew)_{t} + Fw.$$

Integrating be parts we obtain

$$\begin{aligned} & \int_{(-1,1)^2}\!\!\int Lv\left(Lu - \widetilde{g}\right) ds dt \\ = & \int_{(-1,1)^2}\!\!\int vL^t\left(Lu - \widetilde{g}\right) ds dt \\ + & \int_{(-1,1)} \left(\left(Av_s + Bv_t + Dv\right)\left(Lu - \widetilde{g}\right) - v\left(A\left(Lu - \widetilde{g}\right)\right)_s\right)(1,t)\, dt \end{aligned}$$

$$
\begin{aligned}
&- \int_{(-1,1)} \left((Av_s + Bv_t + Dv)(Lu - \tilde{g}) - v\left(A(Lu-\tilde{g})\right)_s \right)(-1,t)\, dt \\
&+ \int_{(-1,1)} \left((Cv_t + Ev)(Lu - \tilde{g}) - v\left(C(Lu-\tilde{g})\right)_t - v\left(B(Lu-\tilde{g})\right)_s \right)(s,1)\, ds \\
&- \int_{(-1,1)} \left((Cv_t + Ev)(Lu - \tilde{g}) - v\left(C(Lu-\tilde{g})\right)_t - v\left(B(Lu-\tilde{g})\right)_s \right)(s,-1)\, ds.
\end{aligned}
$$

We may write this as

$$
\begin{aligned}
& \int_{(-1,1)^2}\int Lv\,(Lu-\tilde{g})\, dsdt \\
= & \int_{(-1,1)^2}\int vL^t\,(Lu-\tilde{g})\, dsdt \\
+ & \int_{(-1,1)} \left((Av_s + Dv)(Lu - \tilde{g}) - v\left(B(Lu-\tilde{g})\right)_t - v\left(A(Lu-\tilde{g})\right)_s \right)(1,t)\, dt \\
- & \int_{(-1,1)} \left((Av_s + Dv)(Lu - \tilde{g}) - v\left(B(Lu-\tilde{g})\right)_t - v\left(A(Lu-\tilde{g})\right)_s \right)(-1,t)\, dt \\
+ & \; vB\,(Lu-\tilde{g})\,(1,t)|_{-1}^{1} - vB\,(Lu-\tilde{g})\,(-1,t)|_{-1}^{1} \\
+ & \int_{(-1,1)} \left((Cv_t + Ev)(Lu - \tilde{g}) - v\left(B(Lu-\tilde{g})\right)_s - v\left(C(Lu-\tilde{g})\right)_t \right)(s,1)\, ds \\
- & \int_{(-1,1)} \left((Cv_t + Ev)(Lu - \tilde{g}) - v\left(B(Lu-\tilde{g})\right)_s - v\left(C(Lu-\tilde{g})\right)_t \right)(s,-1)\, ds.
\end{aligned}
$$

Now u, v are polynomials of degree N and A, B etc. are polynomials of degree $N-1$; moreover g is a polynomial of degree $2N-1$. Hence all the integrands are polynomials of degree $4N-2$ and so the integral may be exactly evaluated by the Legendre-Gauss-Lobatto quadrature formula with $2N+1$ points. Let $t_0^{2N}, \dots, t_{2N}^{2N}$ represent the $(2N+1)$ quadrature points and $w_0^{2N}, \dots, w_{2N}^{2N}$ the corresponding weights. We shall also denote these points by $s_0^{2N}, \dots, s_{2N}^{2N}$. Let r denote $Lu - \tilde{g}$. Then we may write

$$
\begin{aligned}
\int_{(-1,1)^2}\int Lvr dsdt &= \sum_{i=0}^{2N}\sum_{j=0}^{2N} w_i^{2N} w_j^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(L^t r\right)\left(s_i^{2N}, t_j^{2N}\right) \\
&+ \sum_{j=0}^{2N} w_j^{2N}\left(\sum_{i=0}^{2N} d_{2N,i}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\right) A\left(1, t_j^{2N}\right) r\left(1, t_j^{2N}\right) \\
&+ \sum_{j=0}^{2N} w_j^{2N} v\left(1, t_j^{2N}\right)\left(Dr - (Br)_t - (Ar)_s\right)\left(1, t_j^{2N}\right)
\end{aligned}
$$

$$
\begin{aligned}
& - \sum_{j=0}^{2N} w_j^{2N} \left(\sum_{i=0}^{2N} d_{0,i}^{2N} v\left(s_i^{2N}, t_j^{2N}\right) \right) A\left(-1, t_j^{2N}\right) r\left(-1, t_j^{2N}\right) \\
& - \sum_{j=0}^{2N} w_j^{2N} v\left(-1, t_j^{2N}\right) \left(Dr - (Br)_t - (Ar)_s\right)\left(-1, t_j^{2N}\right) \\
& + \sum_{i=0}^{2N} w_i^{2N} \left(\sum_{j=0}^{2N} d_{2N,j}^{2N} v\left(s_i^{2N}, t_j^{2N}\right) \right) C\left(s_i^{2N}, 1\right) r\left(s_i^{2N}, 1\right) \\
& + \sum_{i=0}^{2N} w_i^{2N} v\left(s_i^{2N}, 1\right) \left(Er - (Br)_s - (Cr)_t\right)\left(s_i^{2N}, 1\right) \\
& - \sum_{i=0}^{2N} w_i^{2N} \left(\sum_{j=0}^{2N} d_{0,j}^{2N} v\left(s_i^{2N}, t_j^{2N}\right) \right) C\left(s_i^{2N}, -1\right) r\left(s_i^{2N}, -1\right) \\
& - \sum_{i=0}^{2N} w_i^{2N} v\left(s_i^{2N}, -1\right) \left(Er - (Br)_s - (Cr)_t\right)\left(s_i^{2N}, -1\right) \\
& + \left(vBr(1,1) - vBr(1,-1) + vBr(-1,-1) - vBr(-1,1)\right).
\end{aligned}
$$

Here the matrix $D^{2N} = d_{i,j}^{2N}$ denotes the differentiation matrix. Thus

$$
\frac{dl}{dt}\left(t_i^{2N}\right) = \sum_{j=0}^{2N} d_{i,j}^{2N} l\left(t_j^{2N}\right)
$$

if l is a polynomial of degree less than or equal to $2N$. We may write this as follows. Recollect that $\widetilde{f}_{i,j}^k$ is a filtered representation of $f_{i,j}^k$. Dropping sub and super scripts $\widetilde{g}$ and g are the representation of $\widetilde{f}$ and f in the variables s and t. Let z denote $\mathcal{L}u - g$. Then

$$
\begin{aligned}
\int_{(-1,1)^2}\int Lv\left(Lu - \widetilde{g}\right) ds dt \; = \; & \sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_i^{2N} w_j^{2N} \mathcal{L}^t z\left(s_i^{2N}, t_j^{2N}\right)\right) \\
& + \sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_j^{2N} d_{2N,i}^{2N} \mathcal{A}\left(1, t_j^{2N}\right) z\left(1, t_j^{2N}\right)\right) \\
& + \sum_{j=0}^{2N} w_j^{2N} v\left(1, t_j^{2N}\right)\left(\left(\mathcal{D}z - (\mathcal{B}z)_t - (\mathcal{A}z)_s\right)\left(1, t_j^{2N}\right)\right) \\
& - \sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_j^{2N} d_{0,i}^{2N} \mathcal{A}\left(-1, t_j^{2N}\right) z\left(-1, t_j^{2N}\right)\right) \\
& - \sum_{j=0}^{2N} w_j^{2N} v\left(-1, t_j^{2N}\right)\left(\left(\mathcal{D}z - (\mathcal{B}z)_t - (\mathcal{A}z)_s\right)\left(-1, t_j^{2N}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_i^{2N} d_{2N,j}^{2N} \mathcal{C}\left(s_i^{2N}, 1\right) z\left(s_i^{2N}, 1\right)\right) \\
&+ \sum_{i=0}^{2N} w_i^{2N} v\left(s_i^{2N}, 1\right)\left(\left(\mathcal{E} z - (\mathcal{B} z)_s - (\mathcal{C} z)_t\right)\left(s_i^{2N}, 1\right)\right) \\
&- \sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N}, t_j^{2N}\right)\left(w_i^{2N} d_{0,j}^{2N} \mathcal{C}\left(s_i^{2N}, -1\right) z\left(s_i^{2N}, -1\right)\right) \\
&- \sum_{i=0}^{2N} w_i^{2N} v\left(s_i^{2N}, -1\right)\left(\left(\mathcal{E} z - (\mathcal{B} z)_s - (\mathcal{C} z)_t\right)\left(s_i^{2N}, -1\right)\right) \\
&+ \left(v(1,1)\,\mathcal{B} z(1,1) - v(1,-1)\,\mathcal{B} z(1,-1)\right. \\
&+ \left. v(-1,-1)\,\mathcal{B} z(-1,-1) - v(-1,1)\,\mathcal{B} z(-1,1)\right).
\end{aligned}
$$

Of course, in writing the above we commit an error. It can be argued as in [19] that this error is spectrally small. In fact if we assume that the boundary of the domain Ω is composed of analytic curves, the coefficients of the differential operator are analytic and the data is analytic then the error committed is exponentially small in N.

Hence there is never any need to filter the coefficients of the differential and boundary operators or the data in any of our computations.

Let U^N denote the vector

$$U^N_{(N+1)i+j+1} = u\left(s_i^N, t_j^N\right) \qquad \text{for } 0 \le i \le N,\ 0 \le j \le N,$$

and let

$$U^{2N}_{(2N+1)i+j+1} = u\left(s_i^{2N}, t_j^{2N}\right) \qquad \text{for } 0 \le i \le 2N,\ 0 \le j \le 2N.$$

Similarly

$$Z^{2N}_{(2N+1)i+j} = \mathcal{L} u\left(s_i^{2N}, t_j^{2N}\right) - g\left(s_i^{2N}, t_j^{2N}\right).$$

Then we may write

$$\int_{(-1,1)^2}\int Lv\left(Lu - \widetilde{g}\right) ds dt = \left(V^{2N}\right)^t RZ^{2N}$$

where R is a matrix such that RZ^{2N} is easily computed.

We now show how to evaluate the boundary terms. For this we have to examine the norm of $H^{1/2}(-1,1)$. Now

$$\|l\|^2_{1/2,(-1,1)} \cong \int_{-1}^{1} l^2(t)\,dt + \int_{-1}^{1}\int_{-1}^{1} \frac{(l(x)-l(y))^2}{(x-y)^2}dxdy.$$

Let $l(t)$ be a polynomial of degree less than or equal to $2N-1$. Then $(l(x)-l(y))/(x-y)$ is polynomial of degree less than or equal to $2N-1$ in x and y. And so we may define

$$\begin{aligned}\|l\|^2_{1/2,(-1,1)} &= \sum_{i=0}^{2N} w_i^{2N} l^2\left(t_i^{2N}\right) \\ &+ \sum_{j=0}^{2N}\sum_{i\neq j,i=0}^{2N} w_i^{2N}w_j^{2N}\left(\frac{l\left(t_i^{2N}\right)-l\left(t_j^{2N}\right)}{t_i^{2N}-t_j^{2N}}\right)^2 + \sum_{i=0}^{2N}\left(w_i^{2N}\right)^2\left(\frac{dl}{dt}\left(t_i^{2N}\right)\right)^2.\end{aligned}$$

Thus there is a symmetric positive definite matrix H^{2N} such that

$$\|l\|^2_{1/2,(-1,1)} = \sum_{i=0}^{2N}\sum_{j=0}^{2N} l\left(t_i^{2N}\right)H_{i,j}^{2N} l\left(t_j^{2N}\right).$$

Now a typical boundary term will be of the form

$$\left\|\left(Pu_s-Qu_t\right)(s,1)-\widetilde{h}(s)\right\|^2_{1/2,(-1,1)}$$

and its variation is given by

$$\sum_{i=0}^{2N}\sum_{j=0}^{2N}\left(\left(Pv_s-Qv_t\right)\left(s_i^{2N},1\right)\right)H_{i,j}^{2N}\left(\left(Pu_s-Qu_t\right)\left(s_j^{2N},1\right)-\widetilde{h}\left(s_j^{2N}\right)\right).$$

So we need to examine the term

$$\sum_{i=0}^{2N}\sum_{j=0}^{2N}\left(\left(Pv_s-Qv_t\right)\left(s_i^{2N},1\right)\right)H_{i,j}^{2N}\left(\left(Pu_s-Qu_t\right)\left(s_j^{2N},1\right)-\widetilde{h}\left(s_j^{2N}\right)\right).$$

Let

$$\sigma_i^{2N} = \sum_{j=0}^{2N} H_{i,j}^{2N}\left(\left(Pu_s-Qu_t\right)\left(s_j^{2N},1\right)-\widetilde{h}\left(s_j^{2N}\right)\right).$$

Then

$$
\begin{aligned}
&\sum_{i=0}^{2N} \left(Pv_s - Qv_t\right)\left(s_i^{2N},1\right)\sigma_i^{2N} \\
= &\sum_{i=0}^{2N}\left(\sum_{j=0}^{2N} d_{i,j}^{2N} v\left(s_j^{2N},1\right)\right)\left(P\left(s_i^{2N},1\right)\sigma_i^{2N}\right) - \sum_{i=0}^{2N}\sum_{j=0}^{2N} Q\left(s_i^{2N},1\right) d_{2N,j}^{2N} v\left(s_i^{2N},t_j^{2N}\right)\sigma_i^{2N} \\
= &\sum_{j=0}^{2N} v\left(s_j^{2N},1\right)\left(\sum_{i=0}^{2N} d_{i,j}^{2N} P\left(s_i^{2N},1\right)\sigma_i^{2N}\right) - \sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N},t_j^{2N}\right)\left(Q\left(s_i^{2N},1\right) d_{2N,j}^{2N}\sigma_i^{2N}\right).
\end{aligned}
$$

Let

$$
Y_i^{2N} = \left(\mathcal{P}u_s - \mathcal{Q}u_t - h\right)\left(s_i^{2N},1\right), \quad X^{2N} = H^{2N}Y^{2N},
$$

where P and Q are filtered representations of $\mathcal{P}$ and $\mathcal{Q}$. Similarly $\widetilde{h}$ is a filtered representation of the actual boundary data h. Then we may write

$$
\begin{aligned}
&\sum_{i=0}^{2N}\sum_{j=0}^{2N} \left(Pv_s - Qv_t\right)\left(s_i^{2N},1\right) H_{i,j}^{2N}\left(\left(Pu_s - Qu_t\right)\left(s_j^{2N},1\right) - \widetilde{h}\left(s_j^{2N}\right)\right) \\
= &\sum_{j=0}^{2N} v\left(s_j^{2N},1\right)\left(\sum_{i=0}^{2N} d_{i,j}^{2N}\mathcal{P}\left(s_i^{2N},1\right) X_i^{2N}\right) - \sum_{i=0}^{2N}\sum_{j=0}^{2N} v\left(s_i^{2N},t_j^{2N}\right)\left(\mathcal{Q}\left(s_i^{2N},1\right) d_{2N,j}^{2N} X_i^{2N}\right) \\
= &\left(V^{2N}\right)^t T X^{2N}.
\end{aligned}
$$

Here T is a $(2N+1)^2 \times (2N+1)$ matrix and TX^{2N} can be easily computed. In writing the above we are again committing an error and this error can be shown to be exponentially small in N once more. Hence there is no need to filter the coefficients of the boundary operators or the data.

Adding all the terms we obtain

$$
\begin{aligned}
&\int_{(-1,1)^2}\int Lv\left(Lu - \widetilde{g}\right) dsdt \\
+ &\sum_{i=0}^{2N}\sum_{j=0}^{2N} \left(Pv_s - Qv_t\right)\left(s_i^{2N},1\right) H_{i,j}^{2N}\left(\left(Pu_s - Qu_t\right)\left(s_j^{2N},1\right) - \widetilde{h}\left(s_j^{2N}\right)\right) + \ldots\ldots + \\
= &\left(V^{2N}\right)^t\left(RZ^{2N} + TX^{2N} + \ldots\right) = \left(V^{2N}\right)^t O^{2N}
\end{aligned}
$$

where

$$O^{2N} = RZ^{2N} + TX^{2N} + \ldots$$

is a $(2N+1)^2$ vector which can be easily computed. Now there exists a matrix G^N such that

$$V^{2N} = G^N V^N.$$

Hence

$$\left(V^{2N}\right)^t O^{2N} = \left(V^N\right)^t \left(\left(G^N\right)^t O^{2N}\right).$$

In [18] it has been shown how $\left(G^N\right)^t O^{2N}$ can be computed. Here we just describe the steps involved and refer the reader to [18] for further details.

Let ρ_k^N be the normalizing factors used in computing the *Discrete Legendre Transform* as

$$\rho_k^N = \begin{cases} 1/\left(k+1/2\right) & \text{if } k < N, \text{ and} \\ 2/N & \text{if } k = N. \end{cases}$$

Let $\{O_{i,j}\}_{0\le i\le 2N, 0\le j\le 2N}$ be as follows

$$O_{i,j} = O^{2N}_{i(2N+1)+j}.$$

Now we perform the following set of operations.

1. Define $O_{i,j} \leftarrow O_{i,j} / w_i^{2N} w_j^{2N}$.

2. Calculate $\{\Delta_{i,j}\}_{0\le i\le 2N, 0\le j\le 2N}$ the Legendre transform of $\{O_{i,j}\}_{0\le i\le 2N, 0\le j\le 2N}$. Define

$$\Delta_{i,j} \leftarrow \rho_i^{2N} \rho_j^{2N} \Delta_{i,j}.$$

3. Calculate $\mu_{i,j} \leftarrow \Delta_{i,j} / \rho_i^N \rho_j^N$, $\quad 0 \le i \le N,\ 0 \le j \le N.$

4. Compute ε, the *inverse Legendre transform* of μ. Define

$$\varepsilon_{i,j} \leftarrow w_i^N w_j^N \varepsilon_{i,j}, \quad 0 \leq i \leq N,\, 0 \leq j \leq N.$$

5. Define a vector J of dimension $(N+1)^2$ as

$$J_{i(N+1)+j} = \varepsilon_{i,j} \quad \text{for } 0 \leq i \leq N,\ 0 \leq j \leq N.$$

Then $J = \left(G^N\right)^t O^{2N}$. Thus we see to compute J we do not need to compute and store any matrices such as the mass and stiffness matrices.

In order to compute $SV - JG$ we need to pass the values of $v_{i,j}^k$ and its derivatives defined on $\partial\Omega_{i,j}^k$, or $\partial\tilde{\Omega}_{i,j}^k$, to its neighboring processors as well as to communicate the corresponding values defined on neighboring processors to the processor on which $v_{i,j}^k$ is defined. Moreover when computing the two scalars required to update the approximate solution and the search direction during one step of the conjugate gradient process some scalars have to be passed from each processor to the root processor. The two scalars are then computed in the root processor and passed to every processor. Thus we see communication between processors is small.

Finally to compute $SV - JG$ requires $O\left(N^3\right)$ time on a parallel computer with N processors since to compute the two dimensional Legendre transform on a processor requires $O\left(N^3\right)$ operations.

3.3 Parallelization Techniques

We define the quadratic form

$$\mathcal{W}^N\left(\left\{v_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k}, \left\{v_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right) \tag{3.2}$$
$$= \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|v_{i,j}^k\left(\tau_k,\theta_k\right)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2 + \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|v_{i,j}^k\left(\xi,\eta\right)\right\|_{2,S}^2.$$

In the same way we may define the quadratic form

$$
\begin{aligned}
(3.3)\quad & \mathcal{V}^N\left(\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k},\left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right) \\
= & \sum_{k=1}^{p}\sum_{i=1}^{I_k}\sum_{j=2}^{N}\left\|\Delta v_{i,j}^k(\tau_k,\theta_k)\right\|_{0,\tilde{\Omega}_{i,j}^k}^2+\sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|\left(L_{i,j}^k\right)^a v_{i,j}^k(\xi,\eta)\right\|_{0,S}^2 \\
+ & \sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\|[v]\|_{0,\gamma_s}^2+\|[(v)_x^a]\|_{1/2,\gamma_s}^2+\left\|\left[(v)_y^a\right]\right\|_{1/2,\gamma_s}^2\right) \\
+ & \sum_{k=1}^{p}\sum_{i=1}^{I_k}\sum_{j=1}^{N-1}\left(\left\|\left[\left(v_{i,j}^k\right)_{\tau_k}\right]\left(\eta_{j+1}^k,\theta_k\right)\right\|_{1/2,\left(\psi_i^k,\psi_{i+1}^k\right)}^2+\left\|\left[v_{i,j}^k\right]\left(\eta_{j+1}^k,\theta_k\right)\right\|_{3/2,\left(\psi_i^k,\psi_{i+1}^k\right)}^2\right) \\
+ & \sum_{k=1}^{p}\sum_{i=1}^{I_k-1}\sum_{j=1}^{N}\left(\left\|\left[\left(v_{i,j}^k\right)_{\theta_k}\right]\left(\tau_k,\psi_{i+1}^k\right)\right\|_{1/2,\left(\eta_j^k,\eta_{j+1}^k\right)}^2+\left\|\left[v_{i,j}^k\right]\left(\tau_k,\psi_{i+1}^k\right)\right\|_{3/2,\left(\eta_j^k,\eta_{j+1}^k\right)}^2\right) \\
+ & \sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\left\|v_{i,N+1}^k(\ln\rho,\theta_k)-v_{i,N}^k(\ln\rho,\theta_k)\right\|_{0,\left(\psi_i^k,\psi_{i+1}^k\right)}^2\right. \\
+ & \left\|\left(v_{i,N+1}^k\right)_{\tau_k}(\ln\rho,\theta_k)-\left(v_{i,N}^k\right)_{\tau_k}(\ln\rho,\theta_k)\right\|_{1/2,\left(\psi_i^k,\psi_{i+1}^k\right)}^2 \\
+ & \left.\left\|\left(v_{i,N+1}^k\right)_{\theta_k}(\ln\rho,\theta_k)-\left(v_{i,N}^k\right)_{\theta_k}(\ln\rho,\theta_k)\right\|_{1/2,\left(\psi_i^k,\psi_{i+1}^k\right)}^2\right) \\
+ & \sum_{k=1}^{p}\sum_{j=1}^{N}\left(\left\|v_{1,j}^k\left(\tau_k,\psi_1^k\right)\right\|_{3/2,\left(\eta_j^k,\eta_{j+1}^k\right)}^2+\left\|v_{I_k,j}^k\left(\tau_k,\psi_{I_k+1}^k\right)\right\|_{3/2,\left(\eta_j^k,\eta_{j+1}^k\right)}^2\right) \\
+ & \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\left(\left\|v_{1,j}^k(0,\eta)\right\|_{0,(0,1)}^2+\left\|\left(v_{1,j}^k\right)_{\sigma_k}^a(0,\eta)\right\|_{1/2,(0,1)}^2\right) \\
+ & \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\left(\left\|v_{I_{k,j},j}^k(1,\eta)\right\|_{0,(0,1)}^2+\left\|\left(v_{I_{k,j},j}^k\right)_{\sigma_{k+1}}^a(1,\eta)\right\|_{1/2,(0,1)}^2\right).
\end{aligned}
$$

Then by Theorem 2.3 we have

$$
\begin{aligned}
(3.4)\qquad & \mathcal{W}^N\left(\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k},\left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right) \\
\leq\ & C(\ln N)^2\,\mathcal{V}^N\left(\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k},\left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right).
\end{aligned}
$$

At the same time it is easy to see that there is constant K such that

$$
\begin{aligned}
(3.5)\quad & K\mathcal{V}^N\left(\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k},\left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right) \\
\leq & \left(\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|v_{i,j}^k(\tau_k,\theta_k)\right\|_{2,\left(\eta_j^k,\eta_{j+1}^k\right)\times\left(\psi_i^k,\psi_{i+1}^k\right)}^2+\sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|v_{i,j}^k(\xi,\eta)\right\|_{2,S}^2\right) \\
= & \mathcal{W}^N\left(\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k},\left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right).
\end{aligned}
$$

Hence we can conclude that the two quadratic forms $\mathcal{W}^N$ and $\mathcal{V}^N$ are *spectrally equivalent* and that there exists a constant K such that

$$
\begin{aligned}
(3.6) \qquad & K\mathcal{V}^N\left(\left\{v_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{v_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right) \\
\leq\ & \mathcal{W}^N\left(\left\{v_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{v_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right) \\
\leq\ & K\left(\ln N\right)^2\mathcal{V}^N\left(\left\{v_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{v_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right).
\end{aligned}
$$

It is clear that if we use $\mathcal{W}^N$ as a preconditioner, then the condition number of the preconditioned system is $O\left(\ln N\right)^2$.

Now $\mathcal{W}^N$ is a block diagonal preconditioner where each block corresponds to the H^2 norm of the spectral element function defined on a particular domain. Let $v_{i,j}^k\left(\xi,\eta\right)$ be the spectral element function defined on the square S to which the domain $\Omega_{i,j}^k$ is mapped. Then $v_{i,j}^k\left(\xi,\eta\right)$ is determined by its values at the points $\left\{\xi_l,\eta_m\right\}_{0\leq l\leq N,0\leq m\leq N}$. Dropping sub and superscripts we order the values of $v\left(\xi_l,\eta_m\right)$ in *lexicographic order* and denote them as v_n for $1\leq n\leq\left(N+1\right)^2$. Now consider the bilinear form $\mathcal{S}^N\left(u,v\right)$ induced by the H^2 norm on S, i.e.

$$\mathcal{S}^N\left(u,u\right)=\left\|u\right\|_{H^2(S)}^2.$$

Then there is a matrix A such that

$$\mathcal{S}^N\left(u,v\right)=\sum_{i=1}^{(N+1)^2}\left(\sum_{j=1}^{(N+1)^2}A_{i,j}u_i\right)v_j.$$

The matrix A can be determined by its columns Ae_i where e_i is a unit vector with a one in its i^{th} place and zero everywhere else.

Now using integration by parts Au can be computed in $O\left(N^3\right)$ operations in exactly the same way as we have computed the residual in the normal equations. If we distribute the $\left(N+1\right)^2$ columns among the N_B processors then the matrix A can be computed in $O\left(N^4\right)$ time steps on a parallel computer since $N_B=O\left(N\right)$. Moreover the L-U factorization of A can be performed in time $O\left(N^5\right)$ and stored on every processor.

Once this has been done the action of the inverse of the matrix on different right hand sides can be computed in $O(N^4)$ time on every processor.

We now come to the important issue of load balancing. Since the dimension of the spectral elements $\left(\left\{u_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{u_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right)$ is $(N_j+1)^2$ for $2 \leq j \leq N$ where $\alpha j \leq N_j \leq N$ and α is a positive constant, hence if we were to assign each $u_{i,j}^k(\tau_k,\theta_k)$ defined on $\widetilde{\Omega}_{i,j}^k$ onto different processors this would cause a severe imbalance in the loads assigned to different processors. Alternatively we could choose $N_j = N$ for all $j \geq 2$ defined on $\widetilde{\Omega}_{i,j}^k$ and $u_{i,j}^k(\xi,\eta)$ defined on S. By doing so we shall be able to achieve perfect load balancing among individual processors, but at the cost of making many of the processors do extra computational work which would not increase the accuracy of the numerical solution substantially.

However we should point out that the other strategy has the drawback that the degree of the polynomials $N_j \sim \alpha j$ is data dependent since α is determined by the data. The strategy of choosing $N_j = N$ is thus more robust and hence is to be preferred as it would apply to the most general class of data.

Finally, since we would need to perform $O(N \log N)$ iterations to obtain the solution to exponential accuracy and every iteration requires time $O(N^4)$, the time required to compute the solution would be $O(N^5 \log N)$.

3.4 Error Estimates

Our treatment of error estimates is very similar to the analysis in [8].

From the results in Section 2.2 we have

$$\begin{aligned}(3.7)\qquad &\int_{\eta_j^k}^{\eta_{j+1}^k}\int_{\psi_i^k}^{\psi_{i+1}^k}\sum_{|\alpha|\leq m}\left|D_{\tau_k}^{\alpha_1}D_{\theta_k}^{\alpha_2}u(\tau_k,\theta_k)\right|^2 d\tau_k d\theta_k \\ &\leq\ C\left(\rho\mu_k^{N+1-j}\right)^{2(1-\beta_k)}\left(d^{m-2}(m-2)!\right)^2\end{aligned}$$

for $2 \leq j \leq N$. Here $0 < \mu_k < 1$.

Now consider a quadrilateral $\Omega^k_{i,j}$ with $j > N$. Let

$$U^k_{i,j}(\xi,\eta) = u\left(X^k_{i,j}(\xi,\eta), Y^k_{i,j}(\xi,\eta)\right)$$

where the mapping

$$M^k_{i,j} = \left(X^k_{i,j}(\xi,\eta), Y^k_{i,j}(\xi,\eta)\right)$$

from S to $\overline{\Omega}^k_{i,j}$ has been defined in [8]. Then the following has been proved in Lemma 5.1 of [8].

$U^k_{i,j}$ is analytic on $\overline{S}$ and for some constants C and d independent of i and k and $|\alpha| = m, m = 1, 2, \ldots$, we have

$$\left|D^\alpha U^k_{i,j}(\xi,\eta)\right| \le C m! d^m \tag{3.8}$$

for all $j > N$.

We are now in a position to prove our main theorem which we state below.

Theorem 3.1 *Let $\left\{v^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}$ be a polynomial in τ_k and θ_k of degree N_j for all i and k where $\alpha j \le N_j \le N$ for some positive α for $j > 2$. (The choice of these constants will become clear in what follows)*
Let $\left\{\left\{w^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{w^k_{i,j}(\xi,\eta)\right\}_{i,j,k}\right\}$ minimize $\mathfrak{r}^N\left\{\left\{v^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{v^k_{i,j}(\xi,\eta)\right\}_{i,j,k}\right\}$ over all $\left\{\left\{v^k_{i,j}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{v^k_{i,j}(\xi,\eta)\right\}_{i,j,k}\right\}$. Then there exist constants C and b such that

$$\begin{aligned}
&\sum_{k=1}^{p}\sum_{j=1}^{N}\sum_{i=1}^{I_k}\left\|w^k_{i,j}(\tau_k,\theta_k) - U^k_{i,j}(\tau_k,\theta_k)\right\|^2_{2,\tilde{\Omega}^k_{i,j}} \\
&+ \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|w^k_{i,j}(\xi,\eta) - U^k_{i,j}(\xi,\eta)\right\|^2_{2,S} \\
&\le Ce^{-bN}.
\end{aligned} \tag{3.9}$$

Here $U_{i,j}^{k}(\xi,\eta) = u\left(M_{i,j}^{k}(\xi,\eta)\right)$. Then using the results on approximation theory in Section 5 of [8] we have that there exists a polynomial $\phi_{i,j}^{k}(\xi,\eta)$ of degree N in each variable separately such that

$$\left\|\left(U_{i,j}^{k}-\phi_{i,j}^{k}\right)\right\|_{l,S}^{2} \leq C_s N^{4+2l-2s}\left\|U_{i,j}^{k}\right\|_{s,S}^{2} \tag{3.10}$$

for $0 \leq l \leq 2$ and all $N > s$, where $C_s = Ce^{2s}$. Hence

$$\left\|U_{i,j}^{k}-\phi_{i,j}^{k}\right\|_{2,S}^{2} \leq C_s N^{-2s+8}\left(Cd^s s!\right)^2 \tag{3.11}$$

for $j > N$.

In the same way $U_{i,j}^{k}(\tau_k,\theta_k) = u(\tau_k,\theta_k)$ for $j \leq N$. Then there exists a polynomial $\phi_{i,j}^{k}(\tau_k,\theta_k)$ of degree N_j in τ_k and θ_k separately such that

$$\left\|\left(U_{i,j}^{k}-\phi_{i,j}^{k}\right)\right\|_{2,\tilde{\Omega}_{i,j}^{k}}^{2} \leq C_s\left(N_j\right)^{-2s+8}\left(C\left(\rho\mu_k^{N+1-j}\right)^{(1-\beta_k)}\left(\lambda_k d\right)^{s-2}(s-2)!\right)^2 \tag{3.12}$$

where

$$\lambda_k = \max\left\{\frac{1}{2}\max_i\left(\Delta\psi_i^k\right), \frac{|\ln\mu_k|}{2}, 1\right\}.$$

We have used (3.7) to obtain the above estimate.

Now

$$\left\|U_{i,1}^{k}(\tau_k,\theta_k)\right\|_{m,\tilde{\Omega}_{i,1}^{k}}^{2} \leq \left(C\left(\rho\mu_k^{N}\right)^{(1-\beta_k)} d^{m-2}(m-2)!\right)^2 \tag{3.13}$$

Hence there exists a constant C such that

$$\left\|U_{i,1}^{k}(\tau_k,\theta_k)\right\|_{2,\tilde{\Omega}_{i,1}^{k}}^{2} \leq \left(C\left(\rho\mu_k^{N}\right)^{(1-\beta_k)}\right)^2 \tag{3.14}$$

for all k and i.

Let $\widetilde{f}_{i,j}^{k}(\xi,\eta)$ be the polynomial of degree $(2N-1)$ in each variable separately which is the orthogonal projection of $f_{i,j}^{k}(\xi,\eta)$ in $H^2(S)$ into the space of polynomials of degree $2N-1$. Then

$$\left\| f_{i,j}^{k} - \widetilde{f}_{i,j}^{k} \right\|_{0,S}^{2} \leq C_t (2N-1)^{-2t+8} \left(Cd^t t! \right)^2 . \tag{3.15}$$

Next, let

$$F(\tau_k,\theta_k) = e^{2\tau_k} f(\tau_k,\theta_k) \quad \text{for } -\infty < \tau_k \leq \ln\rho,\ \psi_1^k \leq \theta_k \leq \psi_{I_k+1}^k .$$

Since f is an analytic function we can show that there exist constants C and d such that

$$\int_{\psi_1^k}^{\psi_{I_k+1}^k} \int_{-\infty}^{\ln\sigma} \sum_{|\alpha|\leq k} \left| D_{\tau_k}^{\alpha_1} D_{\theta_k}^{\alpha_2} F \right|^2 d\tau_k d\theta_k \leq C\sigma^2 \left(d^k k! \right)^2 . \tag{3.16}$$

Now let $\widetilde{F}_{i,j}^{k}(\tau_k,\theta_k)$ denote the polynomial of degree $2N_j-1$ in τ_k and θ_k separately that is the orthogonal projection of $F_{i,j}^{k}(\tau_k,\theta_k)$ in $H^2\left((\eta_j^k,\eta_{j+1}^k) \times (\psi_i^k,\psi_{i+1}^k) \right)$ into the space of polynomials of degree $(2N_j-1)$.

Here

$$F_{i,j}^{k}(\tau_k,\theta_k) = F(\tau_k,\theta_k) \qquad \text{for } \eta_j^k \leq \tau_k \leq \eta_{j+1}^k,\ \psi_i^k \leq \theta_k \leq \psi_{i+1}^k .$$

Then we have

$$\left\| F_{i,j}^{k}(\tau_k,\theta_k) - \widetilde{F}_{i,j}^{k}(\tau_k,\theta_k) \right\|_{0,\widetilde{\Omega}_{i,j}^k}^{2} \leq C_{t_j} (2N_j-1)^{-2t_j+8} \left(C\rho\mu_k^{N+1-j} (\lambda_k d)^{t_j} t_j \right)^2$$

for $j > 2$, where λ_k is as in (3.12).

Finally, we have to examine the error committed in approximating the boundary data $g = (g_1, \cdots, g_p)$.

Now we can conclude that

$$\left\|l_{1,j}^{k}\right\|_{m,\left(\eta_{j}^{k},\eta_{j+1}^{k}\right)}^{2} \leq \left(C\left(\rho\mu_{k}^{N+1-j}\right)^{(1-\beta_k)}\left(d^{m}m!\right)\right)^{2}. \tag{3.17}$$

Let $\widetilde{l}_{1,j}^{k}(\tau_k)$ denote the polynomial of degree $(2N_j-1)$ which is the orthogonal projection of $l_{1,j}^{k}(\tau_k)$ in $H^2\left(\eta_j^k,\eta_{j+1}^k\right)$ into the space of polynomials of degree $(2N_j-1)$. Here $j \geq 2$.

Then we can conclude that

$$\begin{aligned} &\left\|l_{1,j}^{k}-\widetilde{l}_{1,j}^{k}\right\|_{3/2,\left(\eta_{j}^{k},\eta_{j+1}^{k}\right)}^{2} \\ &\leq C_{t_j}\left(2N_j-1\right)^{-2t_j+8}\left(C\left(\rho\mu_{k}^{N+1-j}\right)^{(1-\beta_k)}\left(\lambda_k d\right)^{t_j}t_j!\right)^{2}. \end{aligned} \tag{3.18}$$

Let

$$l_{m}^{k}(\eta) = g\left(X_{m}^{p+1}(0,\eta), Y_{m}^{p+1}(0,\eta)\right)$$

for $0 \leq \eta \leq 1$, for m such that $\Gamma_k \bigcap \partial\Omega_m^{p+1} \neq \phi$.

Let $\widetilde{l}_{m}^{k}(\eta)$ denote the polynomial of degree $(2N-1)$ that is the orthogonal projection of $l_{m}^{k}(\eta)$ in $H^2(0,1)$ into the space of polynomials of degree $(2N-1)$. Then we have

$$\left\|l_{m}^{k}-\widetilde{l}_{m}^{k}\right\|_{3/2,(0,1)}^{2} \leq C_t\left(2N-1\right)^{-2t+8}\left(Cd^{t}t!\right)^{2} \tag{3.19}$$

for $t < 2N-1$.

Now consider the set of functions $\left\{\left\{\phi_{i,j}^{k}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{\phi_{i,j}^{k}(\xi,\eta)\right\}_{i,j,k}\right\}$. We wish to show

$$\mathfrak{r}^{N}\left\{\left\{\phi_{i,j}^{k}(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{\phi_{i,j}^{k}(\xi,\eta)\right\}_{i,j,k}\right\},$$

as defined in (3.1), is exponentially small in N.

Using the estimates we have derived it is easy to see that

$$
\begin{aligned}
(3.20)\quad E_1 &= \sum_{k=1}^{p}\sum_{i=1}^{I_k}\sum_{j=2}^{N}\left|\Delta\phi_{i,j}^k(\tau_k,\theta_k)-\widetilde{F}_{i,j}^k(\tau_k,\theta_k)\right|_{0,\widetilde{\Omega}_{i,j}^k}^2 \\
&+ \sum_{k=1}^{p}\sum_{j=1}^{N-1}\sum_{i=1}^{I_k}\left(\left\|\left[\left(\phi_{i,j}^k\right)_{\tau_k}\right]\left(\eta_{j+1}^k,\theta_k\right)\right\|_{1/2,\left(\psi_i^k,\psi_{i+1}^k\right)}^2\right. \\
&+ \left.\left\|\left[\phi_{i,j}^k\right]\left(\eta_{j+1}^k,\theta_k\right)\right\|_{3/2,\left(\psi_i^k,\psi_{i+1}^k\right)}^2\right) \\
&+ \sum_{k=1}^{p}\sum_{j=1}^{N}\sum_{i=1}^{I_k-1}\left(\left\|\left[\left(\phi_{i,j}^k\right)_{\theta_k}\right]\left(\tau_k,\psi_{i+1}^k\right)\right\|_{1/2,\left(\eta_j^k,\eta_{j+1}^k\right)}^2\right. \\
&+ \left.\left\|\left[\phi_{i,j}^k\right]\left(\tau_k,\psi_{i+1}^k\right)\right\|_{3/2,\left(\eta_j^k,\eta_{j+1}^k\right)}^2\right) \\
&+ \sum_{k=1}^{p}\sum_{j=2}^{N}\left\|\phi_{1,j}^k\left(\tau_k,\psi_1^k\right)-\widetilde{l}_{1,j}^k(\tau_k)\right\|_{3/2,\left(\eta_j^k,\eta_{j+1}^k\right)}^2 \\
&+ \sum_{k=1}^{p}\sum_{j=2}^{N}\left\|\phi_{I_k,j}^k\left(\tau_k,\psi_{I_k+1}^k\right)-\widetilde{l}_{2,j}^{k+1}(\tau_k)\right\|_{3/2,\left(\eta_j^k,\eta_{j+1}^k\right)}^2 \\
&\le \sum_{k=1}^{p}\sum_{j=2}^{N}C_{s_j}N_j^{-2s_j+8}\left(C\left(\rho\mu_k^{N+1-j}\right)^{(1-\beta_k)}(\lambda_k d)^{s_j}(s_j!)\right)^2 \\
&+ \sum_{k=1}^{p}\sum_{j=2}^{N}C_{t_j}(2N_j-1)^{-2t_j+8}\left(C\left(\rho\mu_k^{N+1-j}\right)^{(1-\beta_k)}(\lambda_k d)^{t_j}(t_j!)\right)^2 .
\end{aligned}
$$

It remains then to estimate the terms

$$
\begin{aligned}
E_2 &= \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|\left(L_{i,j}^k\right)^a\phi_{i,j}^k(\xi,\eta)-\widetilde{f}_{i,j}^k(\xi,\eta)\right\|_{0,S}^2 \\
(3.21)\quad &+ \sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\|[\phi]\|_{0,\gamma_s}^2+\|[(\phi)_x^a]\|_{1/2,\gamma_s}^2+\left\|\left[(\phi)_y^a\right]\right\|_{1/2,\gamma_s}^2\right) \\
&+ \left\{\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\left\|\phi_{i,N+1}^k(\ln\rho,\theta_k)-\phi_{i,N}^k(\ln\rho,\theta_k)\right\|_{0,\left(\psi_i^k,\psi_{i+1}^k\right)}^2\right.\right. \\
&+ \left\|\left(\phi_{i,N+1}^k\right)_{\tau_k}(\ln\rho,\theta_k)-\left(\phi_{i,N}^k\right)_{\tau_k}(\ln\rho,\theta_k)\right\|_{1/2,\left(\psi_i^k,\psi_{i+1}^k\right)}^2 \\
&+ \left.\left.\left\|\left(\phi_{i,N+1}^k\right)_{\theta_k}(\ln\rho,\theta_k)-\left(\phi_{i,N}^k\right)_{\theta_k}(\ln\rho,\theta_k)\right\|_{1/2,\left(\psi_i^k,\psi_{i+1}^k\right)}^2\right)\right\}
\end{aligned}
$$

$$+ \sum_{k=1}^{p} \sum_{m,\Gamma_k \bigcap \partial\Omega_m^{p+1} \neq \phi} \left(\left\| \phi_m^{p+1}(0,\eta) - \tilde{l}_m^k(\eta) \right\|_{0,(0,1)}^2 \right.$$
$$\left. + \left\| \left(\phi_m^{p+1}\right)_{\sigma_k}^a (0,\eta) - \left(\tilde{l}_m^k\right)_{\sigma_k}^a (\eta) \right\|_{1/2,(0,1)}^2 \right).$$

Now in Section 3.2 we had let $\mathcal{L}_{i,j}^k$ denote the differential operator with analytic coefficients such that

$$\int_{\Omega_{i,j}^k}\int (\Delta u(x,y))^2 \, dxdy = \int_S\int \left(\mathcal{L}_{i,j}^k u(\xi,\eta)\right)^2 d\xi d\eta$$

where

$$\mathcal{L}_{i,j}^k u(\xi,\eta) = \mathcal{A}_{i,j}^k u_{\xi\xi} + \mathcal{B}_{i,j}^k u_{\xi\eta} + \mathcal{C}_{i,j}^k u_{\eta\eta} + \mathcal{D}_{i,j}^k u_\xi + \mathcal{E}_{i,j}^k u_\eta + \mathcal{F}_{i,j}^k u.$$

We can show as in [8] that there exist constants C and d such that

$$\left| D_\xi^{\alpha_1} D_\eta^{\alpha_2} \mathcal{A}_{i,j}^k \right| \leq C d^m m!$$

for all $(\xi,\eta) \in S$ and $|\alpha| \leq m$. A similar statement holds for all the other coefficients of $\mathcal{L}_{i,j}^k$.

Now

$$\left(L_{i,j}^k\right)^a u(\xi,\eta) = A_{i,j}^k u_{\xi\xi} + B_{i,j}^k u_{\xi\eta} + C_{i,j}^k u_{\eta\eta} + D_{i,j}^k u_\xi + E_{i,j}^k u_\eta + F_{i,j}^k u.$$

Here $A_{i,j}^k$ is the orthogonal projection of $\mathcal{A}_{i,j}^k$ in $H^2(S)$ into the space of polynomials of degree $N-1$. The other coefficients of $L_{i,j}^k$ are similarly obtained. Therefore

$$\left| \mathcal{A}_{i,j}^k - A_{i,j}^k \right| \leq \sqrt{C_s}\,(N-1)^{-s+4}\left(C d^s s!\right). \tag{3.22}$$

The same holds true for all other coefficients.

Now

$$
\begin{aligned}
&\left\| \left(L_{i,j}^k\right)^a \phi_{i,j}^k - \widetilde{f}_{i,j}^k \right\|_{0,S}^2 \\
\leq\ & 3 \left(\left\| \mathcal{L}_{i,j}^k U_{i,j}^k - \left(L_{i,j}^k\right)^a U_{i,j}^k \right\|_{0,S}^2 + \left\| \left(L_{i,j}^k\right)^a U_{i,j}^k - \left(L_{i,j}^k\right)^a \phi_{i,j}^k \right\|_{0,S}^2 + \left\| f_{i,j}^k - \widetilde{f}_{i,j}^k \right\|_{0,S}^2 \right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\left\| \left(L_{i,j}^k\right)^a \phi_{i,j}^k - \widetilde{f}_{i,j}^k \right\|_{0,S}^2 \\
\leq\ & C_s \left((N-1)^{-2s+8} \left(Cd^s s!\right)^2 \right) + C_t \left((2N-1)^{-2t+8} \left(Cd^t t!\right)^2 \right). \qquad (3.23)
\end{aligned}
$$

Finally we show how to estimate

$$
\| [\phi] \|_{0,\gamma_s}^2 + \| [(\phi)_x^a] \|_{1/2,\gamma_s}^2 + \left\| \left[(\phi)_y^a \right] \right\|_{1/2,\gamma_s}^2
$$

for any $\gamma_s \subseteq \Omega^{p+1}$. The other boundary terms can be similarly handled. Then γ_s is a side which is common to Ω_m^{p+1} and Ω_n^{p+1} for some m and n. Let us assume that γ_s is the image of the side $\xi = 1$ of the square S under the mapping M_m^{p+1} and $\xi = 0$ of the square S under the mapping M_n^{p+1}. Then

$$
\begin{aligned}
\| [\phi] \|_{0,\gamma_s}^2 &= \int_0^1 \left(\phi_m^{p+1}(1,\eta) - \phi_n^{p+1}(0,\eta) \right)^2 d\eta \\
&\leq C_s \left(N^{-2s+8} \left(Cd^s s!\right)^2 \right).
\end{aligned}
$$

Now

$$
\begin{aligned}
\| [(\phi)_x^a] \|_{1/2,\gamma_s}^2 &= \Big\| \left(\left(\phi_m^{p+1}\right)_\xi \left(\widehat{\xi}_m^{p+1}\right)_x + \left(\phi_m^{p+1}\right)_\eta \left(\widehat{\eta}_m^{p+1}\right)_x \right)(1,\eta) \\
&- \left(\left(\phi_n^{p+1}\right)_\xi \left(\widehat{\xi}_n^{p+1}\right)_x + \left(\phi_n^{p+1}\right)_\eta \left(\widehat{\eta}_n^{p+1}\right)_x \right)(0,\eta) \Big\|_{1/2,(0,1)}^2.
\end{aligned}
$$

We have

$$
\left(U_m^{p+1}\right)_x(1,\eta) = \left(\left(U_m^{p+1}\right)_\xi \left(\xi_m^{p+1}\right)_x + \left(U_m^{p+1}\right)_\eta \left(\eta_m^{p+1}\right)_x \right)(1,\eta)
$$

and

$$\left(U_n^{p+1}\right)_x(0,\eta) = \left(\left(U_n^{p+1}\right)_\xi \left(\xi_n^{p+1}\right)_x + \left(U_n^{p+1}\right)_\eta \left(\eta_n^{p+1}\right)_x\right)(0,\eta).$$

Moreover

$$\left(U_m^{p+1}\right)_x(1,\eta) = \left(U_n^{p+1}\right)_x(0,\eta)$$

and

$$\|ab\|_{1/2,(0,1)} \le \|a\|_{1,\infty,(0,1)} \bullet \|b\|_{1/2,(0,1)}.$$

Now $(\xi_m^{p+1})_x, (\eta_m^{p+1})_x, (\xi_m^{p+1})_y, (\eta_m^{p+1})_y$ are analytic functions of ξ and η on S and satisfy

$$\left|D_\xi^{\alpha_1} D_\eta^{\alpha_2}\left(\left(\xi_m^{p+1}\right)_x\right)\right| \le C d^m m!$$

for $(\xi,\eta) \in S$ and $|\alpha| \le m$. So we can show that

$$\left\|\left(\left(\xi_m^{p+1}\right)_x - \left(\widehat{\xi_m^{p+1}}\right)_x\right)\right\|_{1/2,(0,1)}^2 \le C_s\left(N^{-2s+8}\left(Cd^s s!\right)^2\right) \tag{3.24}$$

and

$$\left|\left(\widehat{\xi_m^{p+1}}\right)_x\right|_{1,\infty,(0,1)}^2 \le C \ln N$$

using Sobolev's embedding theorem and (2.47). Putting all these estimates together we can conclude that

$$\|[(\phi)_x^a]\|_{1/2,\gamma_s}^2 \le C_s\left(N^{-2s+8} \ln N \left(Cd^s s!\right)^2\right). \tag{3.25}$$

Similarly we can show that

$$\left\|\left[(\phi)_y^a\right]\right\|_{1/2,\gamma_s}^2 \le C_s\left(N^{-2s+8} \ln N \left(Cd^s s!\right)^2\right). \tag{3.26}$$

Hence we can conclude that the term E_2, defined in (3.21) satisfies

$$E_2 \leq CpI\left(C_s\left(N^{-2s+8}\ln N\left(Cd^s s!\right)^2\right) + C_t\left(2N-1\right)^{-2t+8}\left(Cd^t t!\right)^2\right). \tag{3.27}$$

We now use Sterling's formula

$$n! \sim \sqrt{2\pi n}e^{-n}n^n$$

to simplify this expression. We choose

$$\begin{aligned} \alpha j &\leq N_j \leq \beta N, \qquad \text{where } 0 < \alpha \text{ and } \beta \leq 1, \\ s_j &= \gamma N_j, \qquad \text{where } 0 < \gamma < 1, \text{ and} \\ t_j &= \gamma\left(2N_j - 1\right). \end{aligned}$$

Moreover $s = \gamma N$ and $t = \gamma\left(2N-1\right)$. Then we obtain

$$\begin{aligned} &\mathfrak{r}^N\left(\left\{\phi_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k}, \left\{\phi_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right) \\ \leq\ &\sum_{k=1}^{p}\sum_{j=2}^{N} C\left(N_j\right)^8\left(\rho\mu_k^{N+1-j}\right)^{2(1-\beta_k)}\left(2\pi\gamma N_j\left(\lambda_k d\gamma\right)^{2\gamma N_j}\right) \\ +\ &I\left(\sum_{k=1}^{p}\sum_{j=2}^{N} C\left(2N_j-1\right)^8\left(\rho\mu_k^{N+1-j}\right)^{2(1-\beta_k)}\left(2\pi\gamma\left(2N_j-1\right)\left(\lambda_k d\gamma\right)^{2\gamma(2N_j-1)}\right)\right) \\ +\ &CpI\left(N^8\ln N\left(2\pi\gamma N\right)\left(d\gamma\right)^{2\gamma N} + \left(2N-1\right)^8\left(2\pi\gamma\left(2N-1\right)\right)\left(d\gamma\right)^{2\gamma(2N-1)}\right). \end{aligned}$$

Select γ so that $(\lambda_k d\gamma) < 1$ for all k. Then there exists a constant $b > 0$ such that the estimate

$$\mathfrak{r}^N\left(\left\{\phi_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k}, \left\{\phi_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right) \leq Ce^{-bN} \tag{3.28}$$

holds.

Let $\left\{\left\{w_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k}, \left\{w_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right\}$ minimize $\mathfrak{r}^N\left\{\left\{v_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k}, \left\{v_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right\}$

over all $\left\{\{v_{i,j}^k(\tau_k,\theta_k)\}_{i,j,k}, \{v_{i,j}^k(\xi,\eta)\}_{i,j,k}\right\}$. Then by (3.28)

$$\mathfrak{r}^N\left(\{w_{i,j}^k(\tau_k,\theta_k)\}_{i,j,k}, \{w_{i,j}^k(\xi,\eta)\}_{i,j,k}\right) \leq Ce^{-bN}. \tag{3.29}$$

Therefore we can conclude that

$$\mathcal{V}^N\left(\{\phi_{i,j}^k(\tau_k,\theta_k) - w_{i,j}^k(\tau_k,\theta_k)\}_{i,j,k}, \{\phi_{i,j}^k(\xi,\eta) - w_{i,j}^k(\xi,\eta)\}_{i,j,k}\right) \leq Ce^{-bN} \tag{3.30}$$

where the functional $\mathcal{V}^N$ has been defined in (3.3).

Hence using the stability theorem 2.3 we obtain

$$\begin{aligned} &\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\phi_{i,j}^k(\tau_k,\theta_k) - w_{i,j}^k(\tau_k,\theta_k)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2 \\ &+ \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|\phi_{i,j}^k(\xi,\eta) - w_{i,j}^k(\xi,\eta)\right\|_{2,S}^2 \\ &\leq Ce^{-bN}. \end{aligned} \tag{3.31}$$

It is easy to show that

$$\begin{aligned} &\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\phi_{i,j}^k(\tau_k,\theta_k) - U_{i,j}^k(\tau_k,\theta_k)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2 \\ &+ \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|\phi_{i,j}^k(\xi,\eta) - U_{i,j}^k(\xi,\eta)\right\|_{2,S}^2 \\ &\leq Ce^{-bN}. \end{aligned} \tag{3.32}$$

And using the above estimates along with (3.14) we obtain

$$\begin{aligned} &\sum_{k=1}^{p}\sum_{j=1}^{N}\sum_{i=1}^{I_k}\left\|U_{i,j}^k(\tau_k,\theta_k) - w_{i,j}^k(\tau_k,\theta_k)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2 \\ &+ \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|U_{i,j}^k(\xi,\eta) - w_{i,j}^k(\xi,\eta)\right\|_{2,S}^2 \\ &\leq Ce^{-bN}, \end{aligned} \tag{3.33}$$

which is the desired result (3.9).□

<u>**Remarks**</u> : It is easy to show that we can define a corrected version of the spectral element solution so that it is conforming and an exponentially accurate solution in the $H^1(\Omega)$ norm and this is described in the next section.

3.5 Computational Results

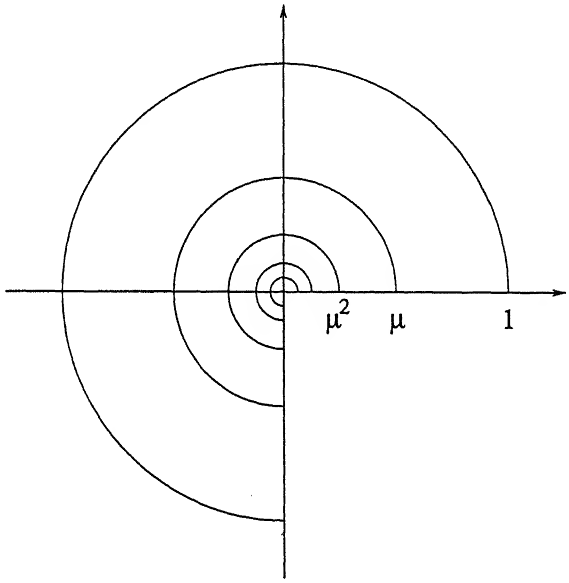

Figure 3.1: The geometric mesh

To verify the asymptotic nature of the results we have obtained we consider Poisson's equation on a polygonal domain with Dirichlet boundary conditions. We consider only a sectoral domain as shown in Fig. 3.1 and show that the mesh chosen (geometric, with ratio .15 which leads to optimal convergence) with Dirichlet boundary data gives exponential convergence.

After having obtained the non-conforming solution on all the elements by the techniques described in previous sections we can make a correction to the spectral element functions $\{u_{i,j}^k(\xi,\eta)\}_{i,j,k}$ so that the corrected spectral element functions $\{\widehat{u}_{i,j}^k(\xi,\eta)\}_{i,j,k}$ are conforming and the error between the exact solution and the corrected approximation in the $H^1(\Omega)$ norm is exponentially small in N. We do this in two steps:

1. First, we make a bilinear correction $\left\{\alpha_{i,j}^{k}(\xi,\eta)\right\}_{i,j,k}$ so that $\left\{\left(u_{i,j}^{k}+\alpha_{i,j}^{k}\right)(\xi,\eta)\right\}_{i,j,k}$ are continuous at the vertices of the rectangles on which they are defined. For this we define $\alpha_{i,1}^{k}\equiv 0$ for all i and k. If P is a vertex of $\Omega_{i,j}^{k}$ for $j\geq 2$ and $P\notin\partial\Omega_{i,1}^{k}$ then we choose $\alpha_{i,j}^{k}(P)$ so that $\left(u_{i,j}^{k}+\alpha_{i,j}^{k}\right)(P)=\overline{u}(P)$, where $\overline{u}(P)$ denotes the average of the values of u at P. If however $P\in\partial\Omega_{i,1}^{k}$ we choose $\alpha_{i,j}^{k}(P)$ so that $\left(u_{i,j}^{k}+\alpha_{i,j}^{k}\right)(P)=0$.

2. Next, we make a correction $\left\{\beta_{i,j}^{k}(\xi,\eta)\right\}_{i,j,k}$ so that $\left\{\left(u_{i,j}^{k}+\alpha_{i,j}^{k}+\beta_{i,j}^{k}\right)(\xi,\eta)\right\}_{i,j,k}$ are conforming. Once again, we define $\beta_{i,1}^{k}\equiv 0$ for all i and k. If γ is a side of $\Omega_{i,j}^{k}$ for $j\geq 2$ and $\gamma\subseteq\partial\Omega_{i,1}^{k}$ then we choose $\beta_{i,j}^{k}$ so that $\left(u_{i,j}^{k}+\alpha_{i,j}^{k}+\beta_{i,j}^{k}\right)(P)=0$ for $P\in\gamma$. Otherwise we choose $\beta_{i,j}^{k}$ so that $\left(u_{i,j}^{k}+\alpha_{i,j}^{k}+\beta_{i,j}^{k}\right)(P)=(\overline{u}+\overline{\alpha})(P)$ for $P\in\gamma$. Now $\beta_{i,j}^{k}$ has it's traces defined on the sides of the square $\mathcal{S}$. Moreover the traces of $\beta_{i,j}^{k}$ are polynomials on the sides of $\mathcal{S}$. Let

$$\begin{aligned}\beta_{i,j}^{k}(\xi,-1) &= \phi_1(\xi),\\ \beta_{i,j}^{k}(\xi,1)) &= \phi_3(\xi),\\ \beta_{i,j}^{k}(-1,\eta) &= \phi_4(\eta),\ \text{and}\\ \beta_{i,j}^{k}(1,\eta) &= \phi_2(\eta).\end{aligned}$$

We then define a lifting of $\beta_{i,j}^{k}(\xi,\eta)$ onto $\mathcal{S}$ as follows

$$\beta_{i,j}^{k}(\xi,\eta)=\frac{1}{2}\left(\phi_1(\xi)(1-\eta)+\phi_3(\xi)(1+\eta)+\phi_2(\eta)(1+\xi)+\phi_4(\eta)(1-\xi)\right).$$

We now define the corrected set of spectral element functions as

$$\widehat{u}_{i,j}^{k}(\xi,\eta)=\left(u_{i,j}^{k}+\alpha_{i,j}^{k}+\beta_{i,j}^{k}\right)(\xi,\eta)\qquad\text{for }(\xi,\eta)\in\mathcal{S}.$$

We present the results of our numerical simulations for a sector with sectoral angle $\omega=\frac{3\pi}{2}$ and radius $\rho=1$. We choose our data so that the solution has the form of

the leading singular solution $u = r^{\alpha} \sin(\alpha\theta)$ where $\alpha = \frac{\pi}{\omega}$. We divide the sector into three equal subsectors and choose the geometric ratio $\mu = .15$. Let N be the number of spectral elements in the radial direction and the number of degrees of freedom of each variable in every element. Table 3.1 shows the percentage of relative error $||e||_{ER}$

Table 3.1: Relative error in percent against N

N	$\|\|e\|\|_{ER}$ %
2	.7462E+01
3	.3709E+01
4	.1407E+01
5	.5151E+00
6	.1736E+00
7	.5720E-01
8	.1830E-01
9	.5779E-02
10	.1798E-02
11	.5547E-03
12	.1696E-03

in the energy norm, defined as $||e||_{ER} = ||e||_E \,/\, ||u||_E$ where $||.||_E$ stands for the energy norm, against N for $\mu = 0.15$. Fig. 3.2 shows the Log of relative error in the energy

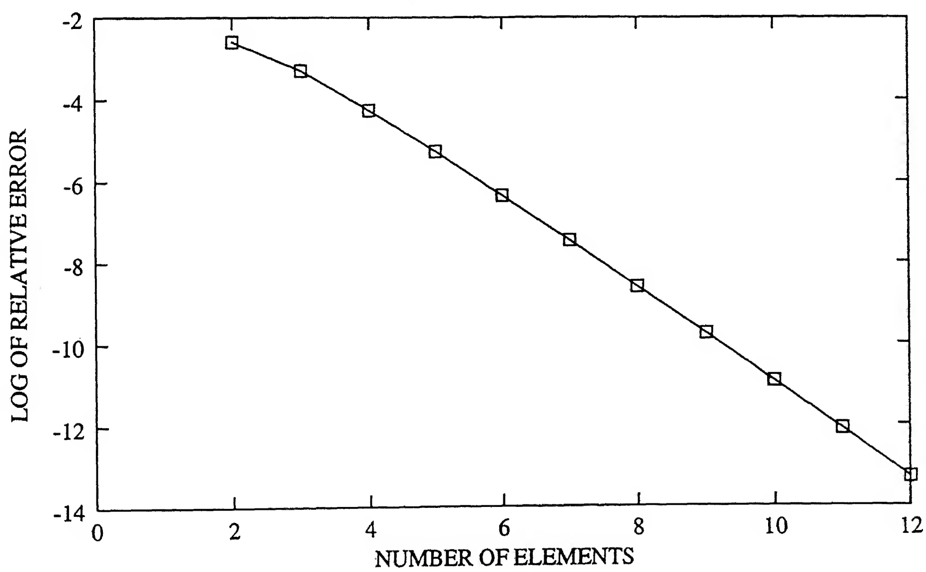

Figure 3.2: Log of relative error vs. N

norm on the scale $\ln ||e||_{ER}$ against N and the relationship is almost linear.

We next choose an analytic solution to examine how the error depends on number

of iterations. Table 3.2 shows the percentage of relative error $||e||_{ER}$ in the energy nor

Table 3.2: Relative error in percent against the number of iterations

Iterations	$\|\|e\|\|_{ER}$ %
10	.8411E+00
20	.2104E+00
30	.4151E-01
40	.7066E-02
50	.1264E-02
60	.2016E-03
70	.6389E-05
80	.8497E-07
90	.7693E-07
100	.7687E-07
112	.7687E-07

against the number of iterations. Fig. 3.3 shows the Log of relative error in the energy norm on the scale $\ln ||e||_{ER}$ against the number of iterations.

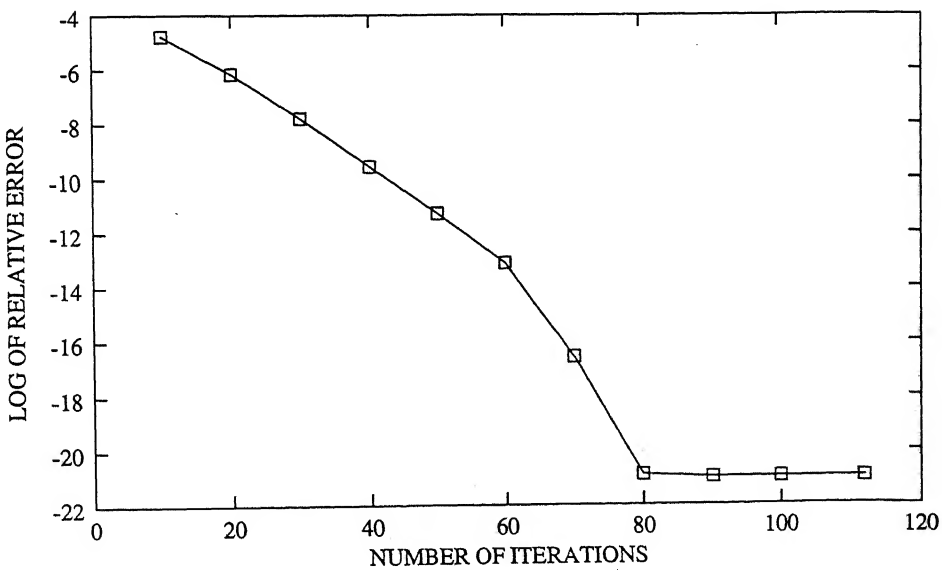

Figure 3.3: Log of relative error vs. the number of iterations

Chapter 4

Mixed Boundary Conditions

4.1 Introduction

In this chapter we continue to work with a set of local coordinate systems, which we shall refer to as *modified polar coordinates* and which were first introduced by Kondratiev in [30], in a sectoral neighborhood of every corner and a global coordinate system elsewhere. We first derive *differentiability estimates* for the solution of the elliptic boundary value problem with mixed boundary conditions similar to those in Chapter 2.

We next prove a *stability theorem* for an essentially *non-conforming* spectral element representation, viz spectral element functions which are non-conforming except that their values are the same at the vertices of the elements on which they are defined. Unfortunately the ***Constant*** in this stability theorem degrades severely compared to the ***Constant*** we obtain for the stability theorem 2.3 which deals with the case of Dirichlet boundary conditions. We now seek a solution to the elliptic boundary problem as in Chapter 3 which minimizes the sum of the squares of weighted L^2 norms of the residuals in the partial differential equation and appropriate *fractional Sobolev norms* of the residuals in the boundary conditions and enforce continuity by adding a term which measures the sum of the squares in the jump of the function and its

derivatives at *inter-element* boundaries in an appropriate fractional Sobolev norm to the functional being minimized. If we were to substitute zero data into this functional then it would exactly correspond to the quadratic form in the right hand side of the stability theorem we have obtained.

To solve the minimization problem we need to solve the *normal equations* which arise from the *least-squares problem* corresponding to the functional we are minimizing. We have seen in Section 3.2 that the residual in the normal equations can be computed cheaply and efficiently without having to compute any *mass and stiffness matrices.*

We now come to the aspect of *parallelization.* One important difference between this chapter and the earlier ones is that the spectral elements $\{u^k_{i,j}\}$ we use are not fully non-conforming in the sense that all the function elements have the same value at common vertices of $\{\Omega^k_{i,j}\}$. The function at the corner most elements of a vertex A_k is taken to be a constant. We divide the vector composed of the values of the spectral element functions at the *Legendre-Gauss-Lobatto* points into two sub vectors - one consisting of the common values of the spectral elements at the vertices of all the elements and the other consisting of the remaining values. The dimension of the first sub vector, which can be thought of as a vector of *common boundary values*, is small compared to FEM.

To solve the system of normal equations we first need to solve a much smaller system of equations corresponding to the *Schur complement* of the sub vector of common boundary values. However to compute the residual for the system of equations we obtain for the Schur complement of the common boundary values we need to be able to compute the action of the inverse of the matrix corresponding to a restricted version of the quadratic form in the right hand side of the stability theorem on any given vector. The common boundary values are the common values of $\{u^k_{i,j}\}_{j\geq 2}$ at the vertices of $\{\Omega^k_{i,j}\}_{j\geq 2}$ and the k different values of $\{u^k_{i,1}\}$ which is the same constant for all i and for a fixed k. This quadratic form also corresponds to the functional we minimize with zero data, restricted to those spectral element functions which vanish at the vertices

of all the elements $\{\Omega^k_{i,j}\}_{j\geq 2}$ and are zero for $\{\Omega^k_{i,1}\}$ for all i and k. Now we are able to prove a stability theorem for precisely this quadratic form, restricted to the space of spectral element functions which vanish at the vertices of all the elements and the ***Constant*** in this stability theorem is as good as that of the stability theorem 2.3.

Hence the parallelization techniques of Chapter 3 immediately apply and we can precondition the matrix corresponding to this restricted quadratic form by a block diagonal matrix with the same matrix occurring repeatedly as its block diagonal elements. An approximate inverse for this matrix can be computed, as in Chapter 3. Thus we can compute the action of the block diagonal matrix acting on a given vector to *exponential accuracy* in N, the number of elements and the number of degrees of freedom in each variable of spectral element functions, using $O\left(N\left(\log N\right)\right)$ iterations of the *preconditioned conjugate gradient method.*

We can therefore solve the system of equations corresponding to the Schur complement S of the sub vector of common boundary values provided we can obtain a good approximation S_a to S. Since the dimension of S is so small, i.e. S is a $N_B \times N_B$ matrix where $N_B = O(N)$, we can compute an accurate approximation to S from it's definition and solve the normal equations using only $O\left(N^{3/2}\log N\right)$ iterations of the preconditioned conjugate gradient method. Finally, we show that the spectral element representation we have computed approximates the actual solution to exponential accuracy in a norm which is the sum of the squares of the H^1 norms of functions defined on individual elements. It is possible to make a correction to the approximate solution such that the corrected solution is conforming and the global error between the corrected and actual solution is exponentially small in N in the $H^1(\Omega)$.

4.2 Differentiability Estimates

Let Ω be a polygonal domain with vertices $A_1, A_2, \ldots, A_p$ and corresponding sides $\Gamma_1, \Gamma_2, \ldots, \Gamma_p$ where Γ_i joins the points A_{i-1} and A_i. In addition let the angle subtended

at A_j be ω_j.

In this chapter we shall examine the solution to the problem

$$\Delta u = f \qquad \text{for } (x,y) \in \Omega, \tag{4.1a}$$

with Dirichlet boundary conditions

$$u = g_j \qquad \text{for } (x,y) \in \Gamma_j,\ \Gamma_j \subseteq \Gamma^{[0]}, \tag{4.1b}$$

and Neumann boundary conditions

$$\frac{\partial u}{\partial n} = g_j \qquad \text{for } (x,y) \in \Gamma_j,\ \Gamma_j \subseteq \Gamma^{[1]}. \tag{4.1c}$$

Here $\Gamma^{[0]} \bigcup \Gamma^{[1]} = \partial\Omega$ and we shall assume that $\Gamma^{[0]} \neq \phi$. In particular, we may assume that $\Gamma_1 \subseteq \Gamma^{[0]}$. For the case when $\Gamma^{[0]} = \phi$ the solution is indeterminate up to a constant and so some additional condition has to be specified for a unique solution to exist. $\Gamma^{[0]}$ will be called the Dirichlet part of the boundary and $\Gamma^{[1]}$ the Neumann part of the boundary.

Let $\mathcal{D} = \left\{ j \left| \Gamma_j \subseteq \Gamma^{[0]} \right. \right\}$ and $\mathcal{N} = \left\{ j \left| \Gamma_j \subseteq \Gamma^{[1]} \right. \right\}$. Let (τ_j, θ_j) denote the modified polar coordinates centered at the vertex A_j. Let

$$a_j = u\left(A_j\right).$$

Then it can be shown as in Chapter 2 that

$$\left\| u\left(\tau_j, \theta_j\right) - a_j \right\|^2_{m,(-\infty,\ln\mu)\times\left(\psi_l^j,\psi_u^j\right)} \leq \mu^{2(1-\beta_j)} \left(C d^{m-2} (m-2)!\right)^2. \tag{4.2}$$

Here $0 < \beta_j < 1$ and $\beta_j > 1 - \pi/\omega_j$ (respectively $\beta_j > 1 - \pi/2\omega_j$ if Dirichlet and Neumann boundary conditions are imposed on the edges Γ_j, Γ_{j+1}, $\overline{\Gamma}_{j+1} \bigcap \overline{\Gamma}_j = A_j$).

4.3 The Numerical Scheme and Stability Estimates

We shall briefly review the introductory portion of Section 2.3 to keep this chapter as self contained as possible.

We first divide Ω into subdomains. Thus we divide Ω into p subdomains $S^1, S^2, \ldots, S^p$, where S^k denotes a domain which contains the vertex A_k and no other, and on each S^k we define a *geometric mesh* as has been done in [8].

Let $\mathfrak{S}^k = \{\Omega_{i,j}^k,\ j = 1, \ldots, J_k,\ i = 1, \ldots, I_{k,j}\}$ be a partition of S^k and let $\mathfrak{S} = \bigcup_{k=1}^p \mathfrak{S}^k$.

In this section we shall derive a stability estimate for a quadratic form, closely related to the functional we will minimize, formulated in terms of an essentially non-conforming spectral element representation of the approximate solution, and which is similar to Theorem 2.3, except that the ***Constant*** in the stability estimate degrades drastically as a function of N.

Such a degradation of the ***Stability Constant*** would cause a drastic degradation in the performance of our parallel schemes. To overcome this problem we make the function representations on different elements continuous at the vertices of the elements only. It is in this sense that the spectral element representations are essentially non-conforming. The method we are going to propose for the solution of the Poisson equation with mixed boundary conditions can therefore be considered as a Vertex Based Method.

The techniques we are going to employ for the proof of the stability theorem are very similar to the ones we use to prove Theorem 2.3, though there are some differences too. We shall therefore highlight only the differences and then simply state the stability theorem, leaving it for the reader to verify it's proof.

Recall that we have defined

$$\Omega_{i,j}^k = \left\{(x,y)\,\middle|\,\sigma_j^k < r_k < \sigma_{j+1}^k,\ \psi_i^k < \theta_k < \psi_{i+1}^k\right\}$$

Let $\tau_k = \ln r_k$. Define

$$\widetilde{\Omega}_{i,j}^k = \left\{(\tau_k, \theta_k) \left| \eta_j^k < \tau_k < \eta_{j+1}^k,\ \psi_i^k < \theta_k < \psi_{i+1}^k \right.\right\} \tag{4.3}$$

for $1 \leq i \leq I_k,\ 1 \leq j \leq N$.

We shall represent $u_{i,1}^k$ in the form

$$u_{i,1}^k (\tau_k, \theta_k) = b_k,\ 1 \leq i \leq I_k, \tag{4.4}$$

where b_k is a constant. We shall represent the remaining $u_{i,j}^k$ as

$$u_{i,j}^k (\tau_k, \theta_k) = \sum_{n=1}^{N_j} \sum_{m=1}^{N_j} a_{m,n} \tau_k^m \theta_k^n,\ 1 \leq i \leq I_k,\ 2 \leq j \leq N, \tag{4.5}$$

where N_j will be specified later and $N_j \leq N$ for all j. To proceed further we need to specify the forms of u_l^{p+1}. Recall that u_l^{p+1} corresponds to $u_{i,j}^k$ for some i, j, k with $j > N$. Hence there is a mapping M_l^{p+1}, also denoted $M_{i,j}^k$, with $j > N$ which maps the interior of the unit squares to Ω_l^{p+1}. We let

$$u_l^{p+1} \left(X_l^{p+1} (\xi, \eta), Y_l^{p+1} (\xi, \eta)\right) = \sum_{n=1}^{N} \sum_{m=1}^{N} a_{m,n} \xi^m \eta^n. \tag{4.6}$$

Now

$$\int_{\widetilde{\Omega}_{i,1}^k} \int \left|u_{i,1}^k\right|^2 d\tau_k d\theta_k = \infty$$

unless $b_k = 0$.

We shall therefore change the weights for the corner elements only.

We begin to see why the ***Constant*** in the stability theorem 4.1, we will prove, must degrade so badly when we prove Lemma 4.1, which is an analog of Lemma 2.4:

Lemma 4.1 *Let $w(\tau)$ be a piecewise smooth function of τ for $\tau \in (-\infty, \eta_{N+1}]$ which has discontinuities only at the points $\eta_2, \eta_3, \cdots, \eta_N$ and which is a constant for $\tau \in$*

$(-\infty, \eta_2)$. *Then*

$$\int_{-\infty}^{\eta_2} w^2(\tau)\, e^{\alpha(\tau-\eta_2)} d\tau + \sum_{j=2}^{N} \int_{\eta_j}^{\eta_{j+1}} w^2(\tau)\, d\tau \tag{4.7}$$
$$\leq CN^2 \left((\Delta\eta)^2 \left(\sum_{j=2}^{N} \int_{\eta_j^k}^{\eta_{j+1}^k} \left(\frac{dw}{d\tau}\right)^2 d\tau \right) \right.$$
$$\left. + \; \Delta\eta) \left(w^2\left(\eta_{N+1}^-\right) + \sum_{j=2}^{N} \left(w\left(\eta_j^+\right) - w\left(\eta_j^-\right) \right)^2 \right) \right).$$

Here α is a positive constant.

Here $\Delta\eta = \Delta\eta_j$, where $\Delta\eta_j = \eta_{j+1} - \eta_j$ for any $2 \leq j \leq N$. Also

$$w\left(\theta_j^+\right) = \lim_{\theta>\theta_j, \theta\to\theta_j} w(\theta), \quad \text{and}$$
$$w\left(\theta_j^-\right) = \lim_{\theta<\theta_j, \theta\to\theta_j} w(\theta).$$

Define a function $s(\tau)$ as follows:

$$s(\tau) = \begin{cases} w\left(\eta_{N+1}^-\right) & \text{for } \eta_N \leq \tau < \eta_{N+1}, \\ w\left(\eta_{N+1}^-\right) + \sum_{j=k+1}^{N} \left(w\left(\eta_j^-\right) - w\left(\eta_j^+\right) \right) & \text{for } \eta_k \leq \tau < \eta_{k+1},\ 1 \leq k \leq N-1. \end{cases}$$

Then $w(\tau)$ may be written as

$$w(\tau) = h(\tau) + s(\tau),$$

where $h(\tau)$ is a continuous function which is differentiable a.e. Moreover $h(\tau)$ is constant for $-\infty < \tau \leq \eta_2$. Now

$$\int_{\eta_2}^{\eta_{N+1}} w^2(\tau)\, d\tau \leq 2 \left(\int_{\eta_2}^{\eta_{N+1}} h^2(\tau)\, d\tau + \int_{\eta_2}^{\eta_{N+1}} s^2(\tau)\, d\tau \right). \tag{4.8}$$

Clearly

$$h(\tau) = -\int_{\tau}^{\eta_{N+1}^-} \frac{dh}{d\eta} d\eta.$$

since $h\left(\eta_{N+1}^{-}\right)=0$. Hence

$$h^2(\tau)\leq(\tau-\eta_{N+1})\int_{\eta_2}^{\eta_{N+1}}\left(\frac{dh}{d\tau}\right)^2 d\tau.$$

From which we can conclude that

$$\begin{aligned}\int_{\eta_2}^{\eta_{N+1}} h^2(\tau)\,d\tau &\leq \frac{(\eta_{N+1}-\eta_2)^2}{2}\int_{\eta_2}^{\eta_{N+1}}\left(\frac{dh}{d\tau}\right)^2 d\tau\\ &\leq \frac{(N\Delta\eta)^2}{2}\left(\sum_{k=2}^{N}\int_{\eta_k}^{\eta_{k+1}}\left(\frac{dw}{d\tau}\right)^2 d\tau\right).\end{aligned}$$

Now by the Sobolev's embedding theorem

$$h^2(\eta_2)\leq C\int_{\eta_2}^{\eta_{N+1}}\left(h^2(\tau)+\left(\frac{dh}{d\tau}\right)^2\right)d\tau.$$

And

$$\int_{-\infty}^{\eta_2} h^2(\tau)\,e^{\alpha(\tau-\eta_2)}d\tau=\frac{h^2(\eta_2)}{\alpha}.$$

Hence

$$\begin{aligned}\text{(4.9)}\qquad &\int_{-\infty}^{\eta_2} h^2(\tau)\,e^{\alpha(\tau-\eta_2)}d\tau+\int_{\eta_2}^{\eta_{N+1}} h^2(\tau)\,d\tau\\ &\leq\ CN^2(\Delta\eta)^2\left(\sum_{k=2}^{N}\int_{\eta_k}^{\eta_{k+1}}\left(\frac{dw}{d\tau}\right)^2 d\tau\right).\end{aligned}$$

Next

$$\begin{aligned}&\int_{-\infty}^{\eta_2} s^2(\tau)\,e^{\alpha(\tau-\eta_2)}d\tau+\int_{\eta_2}^{\eta_{N+1}} s^2(\tau)\,d\tau\\ &\leq\ w^2\left(\eta_{N+1}^{-}\right)\Delta\eta+\left(\sum_{k=2}^{N-1}(N-k+1)\left(w^2\left(\eta_{N+1}^{-}\right)+\sum_{j=k+1}^{N}\left(w\left(\eta_j^{+}\right)-w\left(\eta_j^{-}\right)\right)^2\right)\Delta\eta\right)\\ &+\ \frac{N}{\alpha}\left(w^2\left(\eta_{N+1}^{-}\right)+\sum_{k=2}^{N}\left(w\left(\eta_k^{+}\right)-w\left(\eta_k^{-}\right)\right)^2\right).\end{aligned}$$

And so we can conclude that

$$\begin{aligned}(4.10)\qquad & \int_{-\infty}^{\eta_2} s^2(\tau)\, e^{\alpha(\tau-\eta_2)} d\tau + \int_{\eta_2}^{\eta_{N+1}} s^2(\tau)\, d\tau \\ \leq\ & CN^2(\Delta\eta)\left(w^2\left(\eta_{N+1}^-\right) + \sum_{k=2}^{N}\left(w\left(\eta_k^+\right) - w\left(\eta_k^-\right)\right)^2\right).\end{aligned}$$

Combining (4.8 - 4.10) we obtain the required result.□

When we compare Lemma 4.1 with Lemma 2.4 we see that the constant multiplying the right hand side of (4.7) grows like $O(N^2)$ unlike the case of (2.79) in Lemma 2.4 where the constant is independent of N. It is this difference in behavior which causes the severe degradation in the ***Constant*** in the stability theorem for elliptic problems with mixed Neumann and Dirichlet boundary conditions which we shall prove at the end of this section.

Next, we need to prove a result similar to a *Poincare inequality* in the subdomain Ω^{p+1} which has Dirichlet boundary conditions prescribed on at least part of its boundary, viz. $\partial\Omega^{p+1}\bigcap\Gamma_1$. Recall that Ω^{p+1} is an open set described in terms of (x,y) coordinates. Moreover $\overline{\Omega}^{p+1}$ is divided into subdomains by arcs γ_s which are the sides of the quadrilateral subdomains Ω_l^{p+1} into which $\overline{\Omega}^{p+1}$ is divided.

Now $\overline{\Omega}^{p+1}$ is divided into p subdomains $\overline{\Omega}^{p+1}\bigcap S^k = T^k$, for $k = 1,\ldots,p$. Each of these subdomains in bounded by the circular arc B_ρ^k with center at A_k and radius ρ, by portions of the sides Γ_k and Γ_{k+1} and lastly by ∂S_c^k which consists of a set of piecewise analytic arcs. We shall denote by $\widetilde{B}_\rho^k$ the image of the circular arcs B_ρ^k in (τ_k,θ_k) coordinates.

Recall that there is an analytic mapping $M_{i,j}^k$ from $\overline{\Omega}_{i,j}^k$ onto S which we write as

$$\begin{aligned} x &= X_{i,,j}^k(\xi,\eta), \\ y &= Y_{i,j}^k(\xi,\eta) \end{aligned}$$

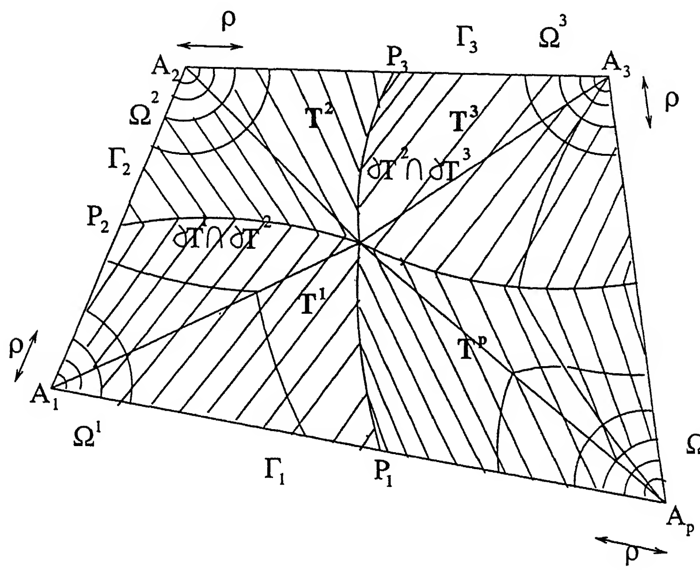

Figure 4.1: Mesh on the whole domain

where $(\xi, \eta) \in S$. By $u_{i,j}^{k}(\xi, \eta)$ we denote the function

$$u_{i,j}^{k}(\xi, \eta) = u_{i,j}^{k}\left(X_{i,j}^{k}(\xi, \eta), Y_{i,j}^{k}(\xi, \eta)\right),$$

and whose form has already been described.

Further we assume that $|\alpha| \leq 2$

$$\text{(4.11a)} \qquad |D^{\alpha}x|, |D^{\alpha}y| \leq C\rho, \qquad \text{and}$$

$$\text{(4.11b)} \qquad C_1\rho^2 \leq J_{i,j}^{k} \leq C^2\rho^2,$$

where $J_{i,j}^{k}$ is the Jacobian of the mapping $M_{i,j}^{k}$ as in (2.9a - 2.9b).

We can now prove Lemma 4.2 which is stated below:

Lemma 4.2 *There exists a constant C, independent of N, such that the following inequality*

$$\text{(4.12)} \quad \sum_{k=1}^{p} \sum_{\gamma_s \subseteq B_\rho^k} \left\| u^k \right\|_{0,\gamma_s}^2 + \sum_{l=1}^{L} \left\| u_l^{p+1}(x,y) \right\|_{0,\Omega_l^{p+1}}^2$$

$$\leq C \left(\sum_{j \in \mathcal{D}} \sum_{\gamma_s \subseteq \partial\Omega^{p+1} \cap \Gamma_j} \left\| u^{p+1} \right\|_{0,\gamma_s}^2 + \sum_{\gamma_s \subseteq \Omega^{p+1}} \left\| \left[u^{p+1} \right] \right\|_{0,\gamma_s}^2 + \sum_{l=1}^{L} \left| u_l^{p+1}(x,y) \right|_{1,\Omega_l^{p+1}}^2 \right)$$

holds.

Now

$$\sum_{i=1}^{I_{1,j}} \int_S \int \left(u_{i,j}^1(\xi,\eta)\right)^2 \left|J_{i,j}^1(\xi,\eta)\right| d\xi d\eta$$
$$\leq C_2 \left\{ \int_0^1 \left(u_{1,j}^1(0,\eta)\right)^2 d\eta + \sum_{i=2}^{I_{1,j}} \int_0^1 \left(u_{i,j}^1(0^+,\eta) - u_{i-1,j}^1(1^-,\eta)\right)^2 d\eta \right.$$
$$\left. + \sum_{i=1}^{I_{1,j}} \int_S \int \left(\left(u_{i,j}^1\right)_\xi (\xi,\eta)\right)^2 d\xi d\eta \right\}.$$

for $j \geq N+1$. Here $J_{i,j}^1(\xi,\eta)$ denotes the Jacobian of the mapping $M_{i,j}^1$. Summing the above over $N+1 \leq j \leq J_1$ we obtain

$$\sum_{j=N+1}^{J_1} \sum_{i=1}^{I_{1,j}} \left\|u_{i,j}^1(x,y)\right\|_{0,\Omega_{i,j}^1}^2 \tag{4.13}$$
$$\leq C \left(\sum_{\gamma_s \subseteq \Gamma_1} \left\|u^1\right\|_{0,\gamma_s}^2 + \sum_{\gamma_s \subseteq T^1} \left\|\left[u^1\right]\right\|_{0,\gamma_s}^2 + \sum_{j=N+1}^{J_1} \sum_{i=1}^{I_{1,j}} \left|u_{i,j}^1(x,y)\right|_{1,\Omega_{i,j}^1}^2 \right).$$

Here as in Chapter 2

$$\left\|u^1\right\|_{0,\gamma_s}^2 = \int_0^1 \left(u_{1,j}^1(0,\eta)\right)^2 d\eta,$$

where $\gamma_s = \partial\Omega_{1,j}^1 \bigcap \Gamma_1$, for $N+1 \leq j \leq J_1$.

The terms $\left\|\left[u^1\right]\right\|_{0,\gamma_s}^2$ are similarly defined for $\gamma_s \subseteq T^1$ (details are similar to that of Chapter 2). The terms

$$\left|u_{i,j}^1(x,y)\right|_{k,\Omega_{i,j}^1}^2$$

are seminorms defined as

$$\left|u_{i,j}^1(x,y)\right|_{k,\Omega_{i,j}^1}^2 = \int_{\Omega_{i,j}^1} \int \sum_{|\alpha|=k} \left|D_x^{\alpha_1} D_y^{\alpha_2} u\right|^2 dxdy.$$

Now from (4.13) we can conclude that

$$\sum_{\gamma_s\subseteq B_\rho^1}\left\|u^1\right\|_{0,\gamma_s}^2+\sum_{\gamma_s\subseteq\partial T^1\bigcap\partial T^2}\left\|u^1\right\|_{0,\gamma_s}^2+\sum_{j=N+1}^{J_1}\sum_{i=1}^{I_{1,j}}\left\|u_{i,j}^1(x,y)\right\|_{0,\Omega_{i,j}^1}^2 \tag{4.14}$$
$$\leq C\left(\sum_{\gamma_s\subseteq\Gamma_1}\left\|u^1\right\|_{0,\gamma_s}^2+\sum_{\gamma_s\subseteq T^1}\left\|\left[u^1\right]\right\|_{0,\gamma_s}^2+\sum_{j=N+1}^{J_1}\sum_{i=1}^{I_{1,j}}\left|u_{i,j}^1(x,y)\right|_{1,\Omega_{i,j}^1}^2\right).$$

In the same way we can show that

$$\sum_{\gamma_s\subseteq B_\rho^2}\left\|u^2\right\|_{0,\gamma_s}^2+\sum_{\gamma_s\subseteq\partial T^2\bigcap\partial T^3}\left\|u^2\right\|_{0,\gamma_s}^2+\sum_{j=N+1}^{J_2}\sum_{i=1}^{I_{2,j}}\left\|u_{i,j}^2(x,y)\right\|_{0,\Omega_{i,j}^2}^2 \tag{4.15}$$
$$\leq C\left(\sum_{\gamma_s\subseteq\partial T^1\bigcap\partial T^2}\left\|u^2\right\|_{0,\gamma_s}^2+\sum_{\gamma_s\subseteq T^2}\left\|\left[u^2\right]\right\|_{0,\gamma_s}^2+\sum_{j=N+1}^{J_2}\sum_{i=1}^{I_{2,j}}\left|u_{i,j}^2(x,y)\right|_{1,\Omega_{i,j}^2}^2\right).$$

Combining (4.14) and (4.15) we obtain

$$\sum_{k=1}^{2}\sum_{\gamma_s\subseteq B_\rho^k}\left\|u^k\right\|_{0,\gamma_s}^2+\sum_{\gamma_s\subseteq\partial T^2\bigcap\partial T^3}\left\|u^2\right\|_{0,\gamma_s}^2+\sum_{\Omega_l^{p+1}\subseteq\overline{T}^1\bigcup\overline{T}^2}\left\|u_l^{p+1}(x,y)\right\|_{0,\Omega_l^{p+1}}^2 \tag{4.16}$$
$$\leq C\left(\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_1}\left\|u^{p+1}\right\|_{0,\gamma_s}^2+\sum_{\gamma_s\subseteq int\left(\overline{T}^1\bigcup\overline{T}^2\right)}\left\|\left[u^{p+1}\right]\right\|_{0,\gamma_s}^2\right.$$
$$\left.+\sum_{\Omega_l^{p+1}\subseteq\overline{T}^1\bigcup\overline{T}^2}\left|u_l^{p+1}(x,y)\right|_{1,\Omega_l^{p+1}}^2\right).$$

Reiterating the argument outlined in (4.13 - 4.16) we immediately conclude that (4.12) holds.□

Next using Lemma 4.1 it is easy to conclude that

$$\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\int_{\widetilde{\Omega}_{i,1}^k}\int\left(u_{i,1}^k(\tau_k,\theta_k)\right)^2e^{\alpha_k\left(\tau_k-\eta_2^k\right)}d\tau_kd\theta_k\right. \tag{4.17}$$
$$\left.+\sum_{j=2}^{N}\int_{\widetilde{\Omega}_{i,j}^k}\int\left(u_{i,j}^k(\tau_k,\theta_k)\right)^2d\tau_kd\theta_k\right)$$
$$\leq CN^2\left\{(\Delta\eta)^2\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\sum_{j=2}^{N}\int_{\widetilde{\Omega}_{i,j}^k}\int\left(u_{i,j}^k\right)_{\tau_k}^2d\tau_kd\theta_k\right)\right.$$
$$\left.+\Delta\eta\sum_{k=1}^{p}\left(\sum_{\widetilde{\gamma}_s\subseteq\widetilde{\Omega}^k,\mu(\widetilde{\gamma}_s)<\infty}\left\|\left[u^k\right]\right\|_{0,\widetilde{\gamma}_s}^2+\sum_{i=1}^{I_k}\left\|u_{i,N}^k(\ln\rho,\theta_k)\right\|_{0,\left(\psi_i^k,\psi_{i+1}^k\right)}^2\right)\right\}.$$

Recall that $\widetilde{\Omega}^k$ is an open set for all k, and $\Delta\eta = \max_{1\leq k\leq p}\{\Delta\eta_k\}$. Moreover $\widetilde{\gamma}_s$ is the image of γ_s in (τ_k, θ_k) coordinates. Combining (4.12) and (4.17) we finally obtain the required estimate

$$\begin{aligned}(4.18)\qquad & \sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\left|u_{i,1}^k\right|^2+\sum_{j=2}^{N}\int_{\widetilde{\Omega}_{i,j}^k}\int\left(\left(u_{i,j}^k\right)(\tau_k,\theta_k)\right)^2 d\tau_k d\theta_k\right)\\
+ & \sum_{l=1}^{L}\left\|u_l^{p+1}(x,y)\right\|_{0,\Omega_l^{p+1}}^2\\
\leq & \; CN^2\left\{\sum_{k=1}^{p}\sum_{i=1}^{I_k}\sum_{j=2}^{N}\int_{\widetilde{\Omega}_{i,j}^k}\int\left(u_{i,j}^k\right)_{\tau_k}^2 d\tau_k d\theta_k+\sum_{l=1}^{L}\left|u_l^{p+1}(x,y)\right|_{1,\Omega_l^{p+1}}^2\right.\\
+ & \sum_{k=1}^{p}\sum_{\widetilde{\gamma}_s\subseteq\widetilde{\Omega}^k,\mu(\widetilde{\gamma}_s)<\infty}\left\|\left[u^k\right]\right\|_{0,\widetilde{\gamma}_s}^2+\sum_{k=1}^{p}\sum_{\widetilde{\gamma}_s\subseteq\widetilde{B}_\rho^k}\left\|\left[u^k\right]\right\|_{0,\widetilde{\gamma}_s}^2\\
+ & \left.\sum_{\gamma_s\subseteq\Omega^{p+1}}\left\|\left[u^{p+1}\right]\right\|_{0,\gamma_s}^2+\sum_{j\in\mathcal{D}}\sum_{\gamma_s\subseteq\Gamma_j\bigcap\partial\Omega^{p+1}}\left\|u^{p+1}\right\|_{0,\gamma_s}^2\right\}.\end{aligned}$$

Now if we combine (4.18), which is the new estimate we have obtained in our treatment of elliptic problems with mixed boundary conditions, with the remaining estimates we have obtained in Section 2.3 we obtain the following stability theorem.

Theorem 4.1 *Consider the domain Ω. Then for N large enough the following stability estimate holds*

$$\begin{aligned}(4.19a)\qquad & \sum_{k=1}^{p}\sum_{i=1}^{I_k}\left|u_{i,1}^k\right|^2+\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|u_{i,j}^k(\tau_k,\theta_k)\right\|_{2,\widetilde{\Omega}_{i,j}^k}^2+\sum_{l=1}^{L}\left\|u_l^{p+1}(\xi,\eta)\right\|_{2,S}^2\\
\leq & \; CN^4\mathcal{V}^N\left(\left\{u_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k},\left\{u_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right).\end{aligned}$$

Here

$$\begin{aligned}(4.19b)\qquad & \mathcal{V}^N\left(\left\{u_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k},\left\{u_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right)\\
= & \left(\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\Delta u_{i,j}^k(\tau_k,\theta_k)\right\|_{0,\widetilde{\Omega}_{i,j}^k}^2\right.\\
+ & \sum_{k=1}^{p}\sum_{\widetilde{\gamma}_l\subseteq\widetilde{\Omega}^k,\mu(\widetilde{\gamma}_l)<\infty}\left(\left\|\left[u^k\right]\right\|_{0,\widetilde{\gamma}_l}^2+\left\|\left[u_{\tau_k}^k\right]\right\|_{1/2,\widetilde{\gamma}_l}^2+\left\|\left[u_{\theta_k}^k\right]\right\|_{0,\widetilde{\gamma}_l}^2\right)\\
+ & \sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\widetilde{\gamma}_s\subseteq\widetilde{\Gamma}_l\bigcap\partial\widetilde{\Omega}^k,\mu(\widetilde{\gamma}_s)<\infty}\left(\left\|u^k\right\|_{0,\widetilde{\gamma}_s}^2+\left\|u_{\tau_k}^k\right\|_{1/2,\widetilde{\gamma}_s}^2\right)\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\tilde{\gamma}_s\subseteq\tilde{\Gamma}_l\bigcap\partial\tilde{\Omega}^k,\mu(\tilde{\gamma}_s)<\infty}\left(\left\|u^k_{\theta_k}\right\|^2_{1/2,\tilde{\gamma}_s}\right)\Bigg) \\
&+ \left(\sum_{k=1}^{p}\sum_{\tilde{\gamma}_l\subseteq\tilde{B}^k_p}\left(\|[u]\|^2_{0,\tilde{\gamma}_l}+\|[u_{\tau_k}]\|^2_{1/2,\tilde{\gamma}_l}+\|[u_{\theta_k}]\|^2_{1/2,\tilde{\gamma}_l}\right)\right) \\
&+ \left(\sum_{l=1}^{L}\left\|\left(\left(L^{p+1}_l\right)^a u^{p+1}_l\right)(\xi,\eta)\right\|^2_{0,S}\right. \\
&+ \sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\|[u]\|^2_{0,\gamma_s}+\|[(u)^a_x]\|^2_{1/2,\gamma_s}+\left\|\left[(u)^a_y\right]\right\|^2_{1/2,\gamma_s}\right) \\
&+ \sum_{k\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_k}\left(\left\|u^{p+1}\right\|^2_{0,\gamma_s}+\left\|\left(u^{p+1}\right)^a_{\sigma_k}\right\|^2_{1/2,\gamma_s}\right) \\
&+ \sum_{k\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_k}\left(\left\|\left(u^{p+1}\right)^a_{\nu_k}\right\|^2_{1/2,\gamma_s}\right)\Bigg).
\end{aligned}
$$

Here $(u^{p+1})^a_{\sigma_k}$ is the approximate tangential derivative and $(u^{p+1})^a_{\nu_k}$ is the approximate outward normal derivative to the side Γ_k as described in Chapter 2. Moreover $[(u)^a_x]$ and $\left[(u)^a_y\right]$ are the approximate jumps in the x and y derivatives of u across γ_s as defined in Chapter 2.

Following the proof of Theorem 2.3 we first show that

$$
\begin{aligned}
&\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left\|u^k_{i,1}e^{\alpha_k\left(\tau_k-\eta^k_2\right)}\right\|^2_{0,\tilde{\Omega}^k_{i,1}}+\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|u^k_{i,j}\left(\tau_k,\theta_k\right)\right\|^2_{2,\tilde{\Omega}^k_{i,j}} \\
&+ \sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|u^k_{i,j}\left(\xi,\eta\right)\right\|^2_{2,S} \\
&\leq CN^4\left(\mathcal{V}^N\left(\left\{u^k_{i,j}\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{u^k_{i,j}\left(\xi,\eta\right)\right\}_{i,j,k}\right)\right).
\end{aligned}
$$

And so this gives us (4.19a).□

Theorem 4.1 gives us a stability estimate for the Poisson equation with mixed boundary condition similar to Theorem 2.3 which deals with the case of Dirichlet boundary conditions. However, unlike Theorem 2.3, the constant multiplying the right hand side of (4.19a) grows rapidly like N^4, where N is the degree of the function elements in each variable and is also proportional to the number of elements into which Ω is divided.

We shall use the estimate (4.19a) to formulate a numerical scheme to obtain an approximate solution which minimizes a functional, closely related to the right hand side of (4.19a), in the remaining portion of this section. However we cannot parallelize our method in an optimal way using (4.19a). To achieve this we shall have to make use of the fact that the spectral elements we have defined are continuous at vertices and this is discussed in Section 4.4.

In the remaining portion of this section we shall formulate our numerical scheme.

Let $f_{i,j}^k = f\left(X_{i,j}^k(\xi,\eta), Y_{i,j}^k(\xi,\eta)\right)$ for $1 \le k \le p$, $N+1 \le j \le J_k$, $1 \le i \le I_{k,j}$, where $(\xi,\eta) \in S$. Let $\widetilde{f}_{i,j}^k(\xi,\eta)$ denote the polynomial of degree $(2N-1)$ in ξ and η, which is the orthogonal projection of $f_{i,j}^k(\xi,\eta)$ into the space of polynomials of degree $(2N-1)$ with respect to the usual inner product on $H^2(S)$. Next, let the vertex $A_k = (x_k, y_k)$. Let $F_{i,j}^k(\tau_k,\theta_k) = e^{2\tau_k} f(x_k + e^{\tau_k}\cos\theta_k, y_k + e^{\tau_k}\sin\theta_k)$, for $1 \le k \le p$, $2 \le j \le N$, $1 \le i \le I_k$. Here $\eta_j^k \le \tau_k \le \eta_{j+1}^k$ and $\psi_i^k \le \theta_k \le \psi_{i+1}^k$.

Consider the case $j \ne 1$. We shall let $\widetilde{F}_{i,j}^k(\tau_k,\theta_k)$ denote the polynomial of degree $(2N_j-1)$ in τ_k and θ_k, which is the orthogonal projection of $F_{i,j}^k(\tau_k,\theta_k)$ into the space of polynomials of degree $(2N_j-1)$ with respect to the usual inner product on $H^2\left(\widetilde{\Omega}_{i,j}^k\right)$.
Now consider the boundary conditions

$u = g_k$ on Γ_k for $k \in \mathcal{D}$, and $\frac{\partial u}{\partial n} = g_k$ on Γ_k for $k \in \mathcal{N}$.

Let

$$l_{1,j}^k(\tau_k,\theta_k) = \begin{cases} u = g_k\left(x_k + e^{\tau_k}\cos\left(\psi_1^k\right), y_k + e^{\tau_k}\sin\left(\psi_1^k\right)\right) \text{ for } k \in \mathcal{D} \\ \frac{\partial u}{\partial \theta_k} = e^{\tau_k} g_k\left(x_k + e^{\tau_k}\cos\left(\psi_1^k\right), y_k + e^{\tau_k}\sin\left(\psi_1^k\right)\right) \text{ for } k \in \mathcal{N}, \end{cases}$$

for $\eta_j^k \le \tau_k \le \eta_{j+1}^k$, where $2 \le j \le N$.

Let $\Gamma_k \bigcap \partial\Omega_j^{p+1} = C_j^k \ne \phi$, be the image of the mapping M_j^{p+1} of S onto $\overline{\Omega}_j^{p+1}$ corresponding to $\xi = 0$.

Let $l_j^k(\eta) = g_k\left(X_j^{p+1}(0,\eta), Y_j^{p+1}(0,\eta)\right)$, where $0 \le \eta \le 1$. We shall let $\widetilde{l}_j^k(\eta)$ denote the polynomial of degree $(2N-1)$ which is the orthogonal projection of $l_j^k(\eta)$ into the space of polynomials of degree $(2N-1)$ with respect to the usual inner product on

$H^2(0,1)$, for $N < j \leq J_k$.

Finally, we have to consider the boundary condition $u = g_k$ on $\Gamma_k \bigcap \partial\Omega^{k-1}$ for $k \in \mathcal{D}$, and $\frac{\partial u}{\partial n} = g_k$ on $\Gamma_k \bigcap \partial\Omega^{k-1}$ for $k \in \mathcal{N}$.

Let

$$l_{2,j}^k(\tau_{k-1}) = \begin{cases} u = g_k\left(x_{k-1} + e^{\tau_{k-1}} \cos\left(\psi_{I_{k-1}+1}^{k-1}\right) : y_{k-1} + e^{\tau_{k-1}} \sin\left(\psi_{I_{k-1}+1}^{k-1}\right)\right) \\ \text{for } k \in \mathcal{D}, \\ \frac{\partial u}{\partial \theta_k} = e^{\tau_{k-1}} g_k\left(x_{k-1} + e^{\tau_{k-1}} \cos\left(\psi_{I_{k-1}+1}^{k-1}\right), y_{k-1} + e^{\tau_{k-1}} \sin\left(\psi_{I_{k-1}+1}^{k-1}\right)\right) \\ \text{for } k \in \mathcal{N}, \end{cases}$$

for $\eta_j^{k-1} \leq \tau_{k-1} \leq \eta_{j+1}^{k-1}$, where $2 \leq j \leq N$.

Now we shall examine how to approximate $l_{i,j}^k$ for $i = 1, 2$ by polynomials of degree $(2N_j - 1)$. For this it is enough to see how to approximate $l_{1,j}^k(\tau_k)$ for each of the cases $k \in \mathcal{D}$ and $k \in \mathcal{N}$.

Here we need restrict ourselves only to $2 \leq j \leq N$. Let us first consider the case $k \in \mathcal{D}$. Now by (4.2) we have

$$\|u(\tau_k, \theta_k) - a_k\|_{m_j,(-\infty,\ln\mu)\times(\psi_l^k,\psi_u^k)}^2 \leq \mu^{2(1-\beta_j)} \left(Cd^{m_j-2}(m_j - 2)!\right)^2.$$

Hence by the trace theorem for Sobolev spaces we have

$$\left\|l_{1,j}^k - a_k\right\|_{m_j,(\eta_j^k,\eta_{j+1}^k)}^2 \leq \left(\rho\mu_k^{N+1-j}\right)^{2(1-\beta_j)} \left(Cd^{m_j-1}(m_j - 1)!\right)^2.$$

Let $g_{1,j}^k(\tau_k)$ be the orthogonal projection of $l_{1,j}^k - a_k$ into the space of polynomials of degree $(2N_j - 1)$ on $H^2\left(\eta_j^k, \eta_{j+1}^k\right)$.

Then using the results on approximation theory in [14] we get

$$\begin{aligned} &\left\|l_{1,j}^k - a_k - g_{1,j}^k\right\|_{2,(\eta_j^k,\eta_{j+1}^k)}^2 \\ \leq\ & C_{m_j}(2N_j - 1)^{-2m_j+8} \left(\rho\mu_k^{N+1-j}\right)^{2(1-\beta_j)} \left(C(\lambda_k d)^{m_j-1}(m_j - 1)!\right)^2, \end{aligned}$$

where

$$\lambda_k = \max\left\{1, \frac{|\ln \mu_k|}{2}, \frac{1}{2}\max_i\left(\Delta\psi_i^k\right)\right\}$$

and $C_s = Ce^s$. Choosing $m_j = \gamma N_j$ with $\frac{1}{2}(\lambda_k d\gamma) < 1$ and using Stirling's formula for approximating $m_j!$ we obtain

$$\left\|l_{1,j}^k - a_k - g_{1,j}^k\right\|_{2,\left(\eta_j^k, \eta_{j+1}^k\right)}^2 \leq Ce^{-bN} \tag{4.20}$$

for some $b > 0$ if we choose $\alpha j < N_j \leq N$ for all $2 \leq j \leq N$. Here α is a positive constant.

Thus we may define $\widetilde{l}_{1,j}^k(\tau_k)$, the polynomial of degree $(2N_j - 1)$ which approximates $l_{1,j}^k(\tau_k)$, to be

$$\widetilde{l}_{1,j}^k(\tau_k) = a_k + g_{1,j}^k(\tau_k)$$

for $\eta_j^k \leq \tau_k \leq \eta_{j+1}^k$. Next, suppose $k \in \mathcal{N}$.

Then by the trace theorem for Sobolev spaces

$$\left\|\frac{\partial u_{1,j}^k}{\partial \theta_k}\left(\tau_k, \psi_1^k\right)\right\|_{m_j,\left(\eta_j^k, \eta_{j+1}^k\right)}^2 \leq \left(\rho\mu_k^{N+1-j}\right)^{2(1-\beta_j)}\left(Cd^{m_j}(m_j)!\right)^2.$$

Hence

$$\left\|l_{1,j}^k(\tau_k)\right\|_{m_j,\left(\eta_j^k, \eta_{j+1}^k\right)}^2 \leq \left(\rho\mu_k^{N+1-j}\right)^{2(1-\beta_j)}\left(Cd^{m_j}(m_j)!\right)^2. \tag{4.21}$$

Let $\widetilde{l}_{1,j}^k(\tau_k)$ denote the orthogonal projection of $l_{1,j}^k(\tau_k)$ into the space of polynomials of degree N_j on $H^1\left(\eta_j^k, \eta_{j+1}^k\right)$. Choosing $m_j = \gamma N_j$ with $\lambda_k d\gamma < 1$ we obtain the estimate

$$\left\|l_{1,j}^k(\tau_k) - \widetilde{l}_{1,j}^k(\tau_k)\right\|_{1,\left(\eta_j^k, \eta_{j+1}^k\right)}^2 \leq Ce^{-bN} \tag{4.22}$$

for some $b > 0$ if we choose $\alpha j < N_j \leq N$. The above estimate applies for Neumann data.

Our numerical scheme may now be formulated as follows:

Find $\left\{ \left\{ u_{i,j}^k \left(\tau_k, \theta_k \right) \right\}_{1 \leq k \leq p, 1 \leq i \leq I_k, 1 \leq j \leq N}, \left\{ u_{i,j}^k \left(\xi, \eta \right) \right\}_{1 \leq k \leq p, 1 \leq i \leq I_{k,j}, N < j \leq J_k} \right\}$

which minimizes the functional

$$
\begin{aligned}
(4.23)\quad & \mathfrak{r}^N \left(\left\{ v_{i,j}^k \left(\tau_k, \theta_k \right) \right\}_{i,j,k}, \left\{ v_{i,j}^k \left(\xi, \eta \right) \right\}_{i,j,k} \right) \\
&= \sum_{k=1}^{p} \sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| \Delta v_{i,j}^k \left(\tau_k, \theta_k \right) - \widetilde{F}_{i,j}^k \left(\tau_k, \theta_k \right) \right\|_{0,\widetilde{\Omega}_{i,j}^k}^2 \\
&+ \sum_{k=1}^{p} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{\Omega}^k, \mu(\widetilde{\gamma}_l) < \infty} \left(\left\| \left[v^k \right] \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| \left[v_{\tau_k}^k \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| \left[v_{\theta_k}^k \right] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \\
&+ \sum_{m \in \mathcal{D}} \sum_{k=m-1}^{m} \sum_{\widetilde{\gamma}_s \subseteq \widetilde{\Gamma}_m \bigcap \partial \widetilde{\Omega}^k, \mu(\widetilde{\gamma}_s) < \infty} \left(\left\| v^k - \widetilde{l}_{m-k+1}^k \right\|_{0,\widetilde{\gamma}_s}^2 + \left\| v_{\tau_k}^k - \left(\widetilde{l}_{m-k+1}^k \right)_{\tau_k} \right\|_{1/2,\widetilde{\gamma}_s}^2 \right) \\
&+ \sum_{m \in \mathcal{N}} \sum_{k=m-1}^{m} \sum_{\widetilde{\gamma}_s \subseteq \widetilde{\Gamma}_m \bigcap \partial \widetilde{\Omega}^k, \mu(\widetilde{\gamma}_s) < \infty} \left\| v_{\theta_k}^k - \widetilde{l}_{m-k+1}^k \right\|_{1/2,\widetilde{\gamma}_s}^2 \\
&+ \sum_{k=1}^{p} \sum_{\widetilde{\gamma}_l \subseteq \widetilde{B}_\rho^k} \left(\left\| [v] \right\|_{0,\widetilde{\gamma}_l}^2 + \left\| [v_{\tau_k}] \right\|_{1/2,\widetilde{\gamma}_l}^2 + \left\| [v_{\theta_k}] \right\|_{1/2,\widetilde{\gamma}_l}^2 \right) \\
&+ \sum_{l=1}^{L} \left\| \left(\left(L_l^{p+1} \right)^a v_l^{p+1} - \widetilde{f}_l^{p+1} \right) (\xi, \eta) \right\|_{0,S}^2 \\
&+ \sum_{\gamma_s \subseteq \Omega^{p+1}} \left(\left\| \left[v^{p+1} \right] \right\|_{0,\gamma_s}^2 + \left\| \left[\left(v^{p+1} \right)_x^a \right] \right\|_{1/2,\gamma_s}^2 + \left\| \left[\left(v^{p+1} \right)_y^a \right] \right\|_{1/2,\gamma_s}^2 \right) \\
&+ \sum_{k \in \mathcal{D}} \sum_{\gamma_s \subseteq \partial \Omega^{p+1} \bigcap \Gamma_k} \left(\left\| v^{p+1} - \widetilde{l}^k \right\|_{0,\gamma_s}^2 + \left\| \left(v^{p+1} \right)_{\sigma_k}^a - \left(\widetilde{l}^k \right)_{\sigma_k}^a \right\|_{1/2,\gamma_s}^2 \right) \\
&+ \sum_{k \in \mathcal{N}} \sum_{\gamma_s \subseteq \partial \Omega^{p+1} \bigcap \Gamma_k} \left(\left\| \left(v^{p+1} \right)_{\nu_k}^a - \left(\widetilde{l}^k \right)_{\nu_k}^a \right\|_{1/2,\gamma_s}^2 \right)
\end{aligned}
$$

over all $\left\{ \left\{ v_{i,j}^k \left(\tau_k, \theta_k \right) \right\}_{i,j,k}, \left\{ v_{i,j}^k \left(\xi, \eta \right) \right\}_{i,j,k} \right\} \in S_V^N$.

As in Chapter 3 it can be shown that

$$
\begin{aligned}
(4.24)\quad & \sum_{k=1}^{p} \sum_{j=1}^{N} \sum_{i=1}^{I_k} \left\| \left(u_{i,j}^k - u \right) (x, y) \right\|_{1,\Omega_{i,j}^k}^2 + \sum_{l=1}^{L} \left\| \left(u_l^{p+1} - u \right) (x, y) \right\|_{1,\Omega_l^{p+1}}^2 \\
& \leq Ce^{-bN}
\end{aligned}
$$

for some constants C and b as $N \to \infty$. In other words, the approximate solution converges to the actual solution at an exponential rate of accuracy in N.

We can make a correction to the spectral element solution so that the corrected solution is conforming and converges to the actual solution at an exponential rate in the $H^1(\Omega)$ norm.

4.4 Parallelization of the Numerical Scheme

Let U denote the values of $\left\{\left\{u_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{u_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\}$ at the Legendre-Gauss-Lobatto quadrature points, except that the common values at the vertices of the quadrilateral elements are counted only once. We may then write U as $U = \begin{bmatrix} U_I \\ U_B \end{bmatrix}$. Here U_B denotes the common values of $\left\{\left\{u_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j\geq 2,k}, \left\{u_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\}$ at the vertices of the quadrilateral elements, and U_I denotes the values of the remaining elements ordered as rows and concatenated in a consistent order of elements as done in Section 3.2.

Let $\mathcal{V}^N\left\{\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\}$ be as defined in (4.19b). Let S_V^N denote the space of functions $\left\{\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\}$ which are continuous at the vertices of the elements on which they are defined. Moreover, let $v_{i,1}^k(\tau_k,\theta_k) = b_k$ for $1 \leq i \leq I_k$. We shall let V denote the values of $\left\{\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j\geq 2,k}, \left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\}$ at the Legendre-Gauss-Lobatto points ordered as at the beginning of this section. Further we may write

$$V = \begin{bmatrix} V_I \\ V_B \end{bmatrix}. \tag{4.25}$$

Then there is a symmetric, positive definite matrix A such that

$$\begin{aligned} &\mathcal{V}^N\left\{\left\{v_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{v_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\} \\ &= \begin{bmatrix} (V_I)^T & (V_B)^T \end{bmatrix} \begin{bmatrix} A_{II} & A_{IB} \\ (A_{IB})^T & A_{BB} \end{bmatrix} \begin{bmatrix} V_I \\ V_B \end{bmatrix}. \end{aligned} \tag{4.26}$$

When we solve the minimization problem (4.23) we have to finally solve a system of equations of the form

$$AU = h, \tag{4.27}$$

where h can also be written as

$$h = \begin{bmatrix} h_I \\ h_B \end{bmatrix}. \tag{4.28}$$

As in Section 3.2 we may replace $\widetilde{f}^k_{i,j}$ by $f^k_{i,j}$ etc. in the data from which we assemble the vector h. In doing so we commit only an exponentially small error as has been argued in Chapter 3. We shall therefore omit the details of how this is done and the interested reader is referred to Section 3.2 for a full description of the steps involved.

Now to solve the system

$$AU = h,$$

we use the block L-U factorization of A, viz.

$$A = \begin{bmatrix} I & 0 \\ A_{IB}^T A_{II}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{II} & 0 \\ 0 & S \end{bmatrix} \begin{bmatrix} I & A_{II}^{-1} A_{IB} \\ 0 & I \end{bmatrix}, \tag{4.29}$$

where the *Schur complement matrix* S is defined as

$$S = A_{BB} - A_{IB}^T A_{II}^{-1} A_{IB}. \tag{4.30}$$

Consequently, solving (4.27) based on the L-U factorization given in (4.29) reduces to solving the system of equations

$$SU_B = \widetilde{h}_B, \quad \text{where} \tag{4.31a}$$

$$\widetilde{h}_B = h_B - A_{IB}^T A_{II}^{-1} h_I, \tag{4.31b}$$

in which the matrix S is the Schur complement matrix described in (4.30).

As we can see from the definition of S the feasibility of such a process depends on our being able to compute $A_{IB}V_B$, $A_{II}V_I$ and $A_{BB}V_B$ for any V_I, V_B cheaply and efficiently, and this can always be done since AV can be computed inexpensively, as has been described in Section 3.2.

However in addition, to this it is imperative that we should be able to construct effective preconditioners for the matrix A_{II}, so that the condition number of the preconditioned system is as small as possible. If we are able to do this it will be possible to compute $(A_{II})^{-1} V_I$ efficiently using the preconditioned conjugate gradient method for any vector V_I. Now consider the space of functions

$$S_0^N = \left\{ \left\{ w_{i,j}^k (\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ w_{i,j}^k (\xi, \eta) \right\}_{i,j,k} \right\} \subseteq S_V^N$$

such that the values of all these functions is zero at all the vertices of the elements

$$\left\{ \left\{ \widetilde{\Omega}_{i,j}^k \right\}_{i,j,k}, \left\{ \left(M_{i,j}^k \right)^{-1} \left(\overline{\Omega}_{i,j}^k \right) = S \right\}_{i,j,k} \right\}$$

on which they are defined. In particular, this implies that $w_{i,1}^k (\tau_k, \theta_k) = 0$ for all i and k. Let W denote the vector of the values of these functions at the Legendre-Gauss-Lobatto points, ordered as at the beginning of this section. Then W has the form

$$W = \begin{bmatrix} W_I \\ 0 \end{bmatrix}, \tag{4.32}$$

and so

$$\mathcal{V}^N \left(\left\{ w_{i,j}^k (\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ w_{i,j}^k (\xi, \eta) \right\}_{i,j,k} \right) = W_I^T A_{II} W_I. \tag{4.33}$$

We now show that for the set of functions

$$\left\{ \left\{ w_{i,j}^k (\tau_k, \theta_k) \right\}_{i,j,k}, \left\{ w_{i,j}^k (\xi, \eta) \right\}_{i,j,k} \right\} \in S_0^N$$

it is possible to define a quadratic form, which consists of a sum of decoupled quadratic forms which is spectrally equivalent to the given quadratic form

$$\mathcal{V}^N \left\{ \left\{ w_{i,j}^k \left(\tau_k, \theta_k \right) \right\}_{i,j,k}, \left\{ w_{i,j}^k \left(\xi, \eta \right) \right\}_{i,j,k} \right\},$$

and the constant of equivalence is at most polylogarithmic in N. We shall define one such quadratic form

$$\mathcal{W}^N \left\{ \left\{ w_{i,j}^k \left(\tau_k, \theta_k \right) \right\}_{i,j,k}, \left\{ w_{i,j}^k \left(\xi, \eta \right) \right\}_{i,j,k} \right\}$$

in what follows, and prove the above assertions we have made for it.

To show this we have to prove a stability theorem similar to Theorem 2.3 for the quadratic form

$$\mathcal{V}^N \left\{ \left\{ w_{i,j}^k \left(\tau_k, \theta_k \right) \right\}_{i,j,k}, \left\{ w_{i,j}^k \left(\xi, \eta \right) \right\}_{i,j,k} \right\}$$

for the case when the function elements

$$\left\{ \left\{ w_{i,j}^k \left(\tau_k, \theta_k \right) \right\}_{i,j,k}, \left\{ w_{i,j}^k \left(\xi, \eta \right) \right\}_{i,j,k} \right\} \in S_0^N,$$

i.e. vanish at the vertices of the elements on which they are defined.

To do so we have to be able to prove a result similar to Lemma 2.2 and this we do in the following lemma.

Lemma 4.3 *Let $w_{i,j}^k \left(\xi, \eta \right)$ be a polynomial of degree N in ξ and η separately, defined on the unit square $S = [0,1] \times [0,1]$, and which is zero at all the vertices of the square. Then there exists a positive constant C such that*

$$\left| w_{i,j}^k \left(\xi, \eta \right) \right|_{0,S}^2 \leq C \left(\left| w_{i,j}^k \left(\xi, \eta \right) \right|_{1,S}^2 + \left| w_{i,j}^k \left(\xi, \eta \right) \right|_{2,S}^2 \right). \tag{4.34}$$

Consider $w_{i,j}^k \left(\xi, \eta \right)$ defined on $[0,1] \times [0,1]$.

Now $w^k_{i,j}(0,0) = 0$. Hence

$$w^k_{i,j}(\xi,0) = \int_0^\xi \frac{\partial w^k_{i,j}}{\partial \xi'}\left(\xi',0\right) d\xi'.$$

And so we can conclude that

$$\left|w^k_{i,j}(\xi,0)\right|^2 \le \xi \int_0^1 \left|\frac{\partial w^k_{i,j}}{\partial \xi}(\xi,0)\right|^2 d\xi.$$

Integrating the above with respect to ξ we obtain

$$\begin{aligned}(4.35)\qquad \int_0^1 \left|w^k_{i,j}(\xi,0)\right|^2 d\xi &\le \frac{1}{2}\int_0^1 \left|\frac{\partial w^k_{i,j}}{\partial \xi}(\xi,0)\right|^2 d\xi \\ &\le K\left(\left|w^k_{i,j}(\xi,\eta)\right|^2_{1,S} + \left|w^k_{i,j}(\xi,\eta)\right|^2_{2,S}\right),\end{aligned}$$

by the trace theorem for Sobolev spaces.

Again

$$w^k_{i,j}(\xi,\eta) = w^k_{i,j}(\xi,0) + \int_0^\eta \frac{\partial w^k_{i,j}}{\partial \eta'}\left(\xi,\eta'\right) d\eta'.$$

Therefore

$$\left|w^k_{i,j}(\xi,\eta)\right|^2 \le 2\left|w^k_{i,j}(\xi,0)\right|^2 + 2\eta \int_0^1 \left|\frac{\partial w^k_{i,j}}{\partial \eta}(\xi,\eta)\right|^2 d\eta.$$

Integrating the above with respect to ξ and η we get

$$\int_S\int \left|w^k_{i,j}(\xi,\eta)\right|^2 d\xi d\eta \le 2\int_0^1 \left|w^k_{i,j}(\xi,0)\right|^2 d\xi + \int_S\int \left|\frac{\partial w^k_{i,j}}{\partial \eta}(\xi,\eta)\right|^2 d\xi d\eta.$$

Combining the above with (4.35) we obtain the required result.□

Clearly Lemma 4.3 applies equally well to any of the function elements $w^k_{i,j}(\tau_k,\theta_k)$ for $2 \le j \le N$, $1 \le i \le I_k$, $1 \le k \le p$, although with a constant C_k which depends on k. Taking the supremum over the constant C_k (as given in (4.34)) we conclude that

$$(4.36)\qquad \left|w^k_{i,j}(\tau_k,\theta_k)\right|^2_{0,\tilde{\Omega}^k_{i,j}} \le C\left(\left|w^k_{i,j}(\tau_k,\theta_k)\right|^2_{1,\tilde{\Omega}^k_{i,j}} + \left|w^k_{i,j}(\tau_k,\theta_k)\right|^2_{2,\tilde{\Omega}^k_{i,j}}\right),$$

for all function elements with $1 \leq k \leq p$, $1 \leq i \leq I_k$, $2 \leq j \leq N$. Here C, of course, denotes a generic constant.

Combining the estimates (4.34) and (4.36) with the remaining estimates we have obtained in Section 2.3 we obtain the following theorem, which in essence, is the same as Theorem 2.3:

Theorem 4.2 *Let $\left\{\left\{w_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{w_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\}$ belong to the space of functions S_0^N which are zero at the vertices of the elements on which they are defined. Then the following estimate holds:*

$$\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|w_{i,j}^k(\tau_k,\theta_k)\right\|_{2,\widetilde{\Omega}_{i,j}^k}^2+\sum_{l=1}^{L}\left\|w_l^{p+1}(\xi,\eta)\right\|_{2,S}^2 \leq C(\ln N)^2\mathcal{V}^N\left(\left\{w_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{w_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right), \tag{4.37}$$

for N large enough.

In the above $w_{i,1}^k(\tau_k,\theta_k)$ is taken to be identically zero for $1 \leq k \leq p$ and $1 \leq i \leq I_k$.□

Now as in Chapter 3 we define a quadratic form

$$\mathcal{W}^N\left(\left\{w_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{w_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right) = \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|w_{i,j}^k(\tau_k,\theta_k)\right\|_{2,\widetilde{\Omega}_{i,j}^k}^2+\sum_{l=1}^{L}\left\|w_l^{p+1}(\xi,\eta)\right\|_{2,S}^2 \tag{4.38}$$

for all $\left\{\left\{w_{i,j}^k(\tau_k,\theta_k)\right\}_{i,j,k}, \left\{w_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\} \in S_0^N$ defined on

$$\left\{\left\{\widetilde{\Omega}_{i,j}^k\right\}_{i,j,k}, \left\{\left(M_{i,j}^k\right)^{-1}\left(\overline{\Omega}_{i,j}^k\right)=S\right\}_{i,j,k}\right\}.$$

Recall that $\widetilde{\Omega}_{i,j}^k$ is a rectangle of the form $\left(\eta_j^k,\eta_{j+1}^k\right)\times\left(\psi_i^k,\psi_{i+1}^k\right)$, and S is the unit square $[0,1]\times[0,1]$.

Now using the trace theorem for Sobolev spaces we have that there exists a constant

K such that

$$
\begin{aligned}
&K\mathcal{V}^N\left(\left\{w_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{w_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right)\\
\leq\ &\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|w_{i,j}^k\left(\tau_k,\theta_k\right)\right\|_{2,\tilde{\Omega}_{i,j}^k}^2+\sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|w_{i,j}^k\left(\xi,\eta\right)\right\|_{2,S}^2\\
=\ &\mathcal{W}^N\left(\left\{w_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{w_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right).
\end{aligned}
\tag{4.39}
$$

Hence we can conclude that the quadratic form $\mathcal{W}^N$ and $\mathcal{V}^N$ are spectrally equivalent and that there exists a constant K such that

$$
\begin{aligned}
&K\mathcal{V}^N\left(\left\{w_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{w_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right)\\
\leq\ &\mathcal{W}^N\left(\left\{w_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{w_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right)\\
\leq\ &C\left(\ln N\right)^2\mathcal{V}^N\left(\left\{w_{i,j}^k\left(\tau_k,\theta_k\right)\right\}_{i,j,k},\left\{w_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right).
\end{aligned}
\tag{4.40}
$$

To balance the loads on all the processors, and to avoid the complication of having to compute α, which is data dependent, we choose $N_j = N$ for $2 \leq j \leq N$ and assign each $u_{i,j}^k$ for $j \geq 2$ on a separate processor. Here α is the constant which has to be chosen such that $N_j \sim \alpha j$ so that the error is exponentially small in N.

We can now use the quadratic form $\mathcal{W}^N$, which consists of a decoupled set of quadratic forms for the solution on each element, as a preconditioner for $\mathcal{V}^N$. We can do this by inverting the block diagonal matrix representation of $\mathcal{W}^N$ since there all the matrices can be replaced by a single matrix of size (N^2+2N-3) by (N^2+2N-3) which occurs repetitively as these blocks. We can assemble this matrix by distributing the computation of its columns among the N_B processors. We can then compute the inverse of this matrix using L-U decomposition distributed among the N_B processors as has been described in Chapter 3.

Thus from (4.40) we can conclude that if we were to compute $(A_{II})^{-1}U_I$ using the preconditioned conjugate gradient method then the condition number of the preconditioned matrix would be $O(\ln N)^2$. Hence to reduce the error in the approximate solution by a factor of e would require $O(\ln N)$ iterations. Thus to compute $(A_{II})^{-1}U_I$

to within an accuracy of $O\left(e^{-bN}\right)$ would require $O(N \ln N)$ iterations of the preconditioned conjugate gradient method.

We now return to the steps involved in solving the system of equations

$$AU = h.$$

As a first step we need to solve the much smaller system of equations:

$$SU_B = \widetilde{h}_B,$$

where

$$\widetilde{h}_B = h_B - A_{IB}^T A_{II}^{-1} h_I$$

and the Schur complement matrix S is defined to be

$$S = A_{BB} - A_{IB}^T A_{II}^{-1} A_{IB}.$$

Now the dimension of the vector U_B is $N_B = O(N)$, which is proportional to the number of elements into which we have divided the original domain Ω. Now to solve (4.31a) to an accuracy of $O\left(e^{-bN}\right)$ we need to be able to compute the residual

$$R_B = SV_B - \widetilde{h}_B \tag{4.41}$$

with the same accuracy and in an efficient manner. Hence we have to be able to compute

$$SV_B = A_{BB}V_B - A_{IB}^T A_{II}^{-1} A_{IB} V_B$$

for any V_B.

As we have already seen $A_{BB}V_B$, $A_{II}V_I$, $A_{IB}V_B$ and $(A_{IB})^T V_I$ can be computed economically and with communication only among neighboring processors since this holds

true when we compute AV for V. Hence the bottleneck in computing R_B consists in computing $A_{II}^{-1}A_{IB}V_B$ to $O\left(e^{-bN}\right)$ and we have seen already that this can be done using $O\left(N\ln N\right)$ iterations of the preconditioned conjugate gradient method for computing $A_{II}^{-1}V_I$ for a given vector V_I.

Thus to compute the residual R_B as defined in (4.41) to an accuracy of $O\left(e^{-bN}\right)$ requires $O\left(N\ln N\right)$ iterations of the preconditioned conjugate gradient method for computing $A_{II}^{-1}\left(A_{IB}V_B\right)$ for a given vector V_B.

We shall now briefly examine the complexity of the solution procedure for the *h-p* version of the FEM. The set of common boundary values for the FEM consists of the values of the spectral element functions at the vertices and sides of the elements. In [26] it has been shown that we can construct an approximation S_a to the Schur complement S such that the condition number κ of the preconditioned system satisfies

$$\kappa \leq C\left(1+(\ln N)^2\right)$$

for a constant C.

Thus to solve (4.31a) to an accuracy of $O\left(e^{-bN}\right)$ will require $O\left(N\log N\right)$ iterations of the preconditioned conjugate gradient method using S_a as a preconditioner. Now to compute the residual in the Schur complement to an accuracy of $O\left(e^{-bN}\right)$ requires $O\left(N\right)$ iterations of the preconditioned conjugate gradient method to compute $\left(A_{II}^{-1}\left(A_{IB}V_B\right)\right)$ [42].

Hence we need to perform $O\left(N^2\log N\right)$ iterations of the preconditioned conjugate gradient method for computing $\left(A_{II}^{-1}\right)V_I$, where V_I will vary after every sequence of $O\left(N\log N\right)$ steps. To solve (4.31a) to an accuracy of $O\left(e^{-bN}\right)$ will require time $O\left(N^6\log N\right)$ on a parallel computer with N processors.

However we now show that it is possible to solve (4.31a) with a complexity which is less than that we have described above.

Recall, to begin with, that S is a $N_B\times N_B$ matrix, where $N_B=O\left(N\right)$. Hence S is matrix of very small dimension compared to the usual size of the Schur complement

matrices that arises in *parallelization techniques* for FEM. Let e^k be a column vector of dimension N_B with a 1 in the k^{th} place and zeroes everywhere else. Let

$$S^k = Se^k.$$

Then $S = \left[\begin{array}{cccc} S^1 & S^2 & \dots & S^{N_B} \end{array} \right]$.
Now using (4.19a) we can show that

$$\|S^{-1}\| \leq CN^4. \tag{4.42}$$

Moreover, it is easy to show that

$$\|A_{IB}\| \leq CN^6. \tag{4.43}$$

Now

$$S^k = A_{BB}e^k - A_{IB}^T A_{II}^{-1} A_{IB} e^k.$$

Suppose we compute an approximation S_a^k to S^k using $O\left(N^{1/2}\log N\right)$ iterations of the preconditioned conjugate gradient method for computing $A_{II}^{-1}V_I$. Then using (4.42) and (4.43) we can show that the error between S_a^k and S^k would be $O\left(e^{-bN^{1/2}}\right)$. If we were to do this for every $1 \leq k \leq N_B$ then the number of iterations required for computing S_a would be $O\left(N^{3/2}\log N\right)$. We would then have computed the matrix S_a, which is an approximate representation of the Schur complement matrix S. In fact, these two matrices would not be just spectrally close but we would have

$$\left(1 - ce^{-bN^{1/2}}\right) S \leq S_a \leq \left(1 + ce^{-bN^{1/2}}\right) S \tag{4.44}$$

for N large enough where b and c are positive constants. Hence if we were to solve (4.31a) using S_a as a preconditioner for S then the error after $N^{1/2}$ steps would be $O\left(e^{-bN}\right)$. Since each step of this process requires $O\left(N\log N\right)$ iterations for computing $\left(A_{II}\right)^{-1}\left(A_{IB}V_B\right)$ we would have performed $O\left(N^{3/2}\log N\right)$ steps of the preconditioned

conjugate gradient method for computing $A_{II}^{-1}(A_{IB}V_B)$, where V_B would vary after $O(N \log N)$ steps, during this process.

This is the same as the number of iterations required to form the approximate representation of the Schur complement, S_a. Hence we can solve (4.31a) to an exponential accuracy in N using only $O\left(N^{3/2}\log N\right)$ iterations of the preconditioned conjugate gradient method for computing $(A_{II})^{-1}W_I$. Thus we have indeed shown that it is possible to solve (4.31a) with a complexity less than that which usual methods require.

Finally, it remains to compute U_I. Now from (4.27) we have

$$A_{II}U_I = h_I - A_{IB}U_B. \tag{4.45}$$

Clearly we can solve the above system of equations for U_I to an accuracy of $O\left(e^{-bN}\right)$ using $O(N\log N)$ iterations of the preconditioned conjugate gradient method to compute $(A_{II})^{-1}V_I$ for a given vector V_I.

Thus the overall complexity of the method we have described would require $O\left(N^{3/2}\log N\right)$ iterations of the preconditioned conjugate gradient method to obtain a solution to the system of equations (4.27) to an exponential accuracy. Moreover the time required on a parallel computer with N processors would be $O\left(N^{5.5}\log N\right)$.

Our treatment of error estimates is similar to the analysis in [8] and in Chapter 3. We do not go into further details to avoid repetition.

4.5 Computational Results

To verify the asymptotic error estimates and estimates of computational complexity we consider Poisson's equation on a polygonal domain with mixed boundary conditions. We consider only a sectoral domain with mixed boundary data and show that our method gives exponential convergence.

The solution we have obtained by the techniques described is conforming only at the vertices. We make a correction to the spectral element functions $\left\{u_{i,j}^k(\xi,\eta)\right\}_{i,j,k}$

as in Chapter 3 so that the corrected spectral element functions $\left\{\widehat{u}_{i,j}^{k}\left(\xi,\eta\right)\right\}_{i,j,k}$ are conforming and the error between the exact solution and the corrected approximation in the $H^1\left(\Omega\right)$ norm is exponentially small in N.

We now present the results of our numerical simulations for a sector with sectoral angle $\omega = \frac{3\pi}{2}$ and radius $\rho = 1$. We choose our data so that the solution has the form of the leading order singular solution $u = r^{\alpha}\cos\left(\alpha\theta\right)$ where $\alpha = \frac{\pi}{\omega}$. We divide the sector into three equal subsectors and choose the geometric ratio $\mu = .15$. The boundary conditions are as follows:

$$\begin{aligned} \left.\frac{\partial u}{\partial n}\right|_{\Gamma_1} &= g_1, \\ \left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} &= g_2, \text{ and} \\ \left.u\right|_{\Gamma_3} &= g_3. \end{aligned}$$

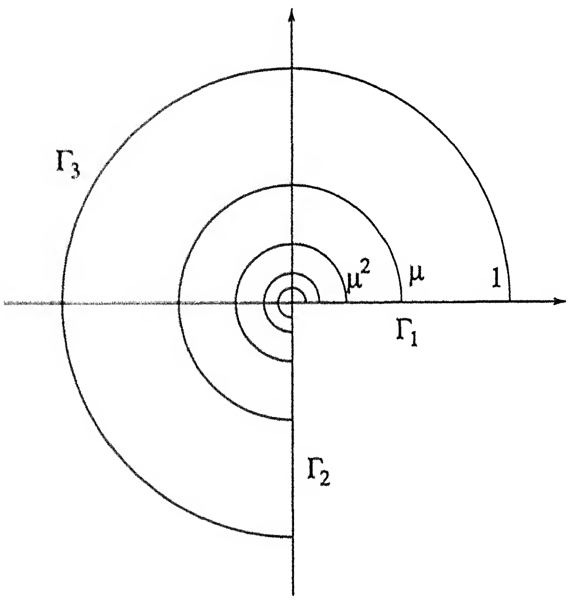

Figure 4.2: The geometric mesh

Let N be the number of spectral elements in the radial direction and the number of degrees of freedom of each variable in every element. Table 4.1 shows the percentage of relative error $\|e\|_{ER}$ in the energy norm, defined as $\|e\|_{ER} = \|e\|_E / \|u\|_E$ where $\|\cdot\|_E$

Table 4.1: Relative error in percent against N

N	$\|\|e\|\|_{ER}$ %
2	.7649E+01
3	.3720E+01
4	.1443E+01
5	.5153E-00
6	.1740E-00
7	.5723E-01
8	.1831E-01
9	.5780E-02
10	.1799E-02
11	.5561E-03
12	.1696E-03

error in the energy norm on the scale $\ln \|e\|_{ER}$ against N and the relationship is almost linear.

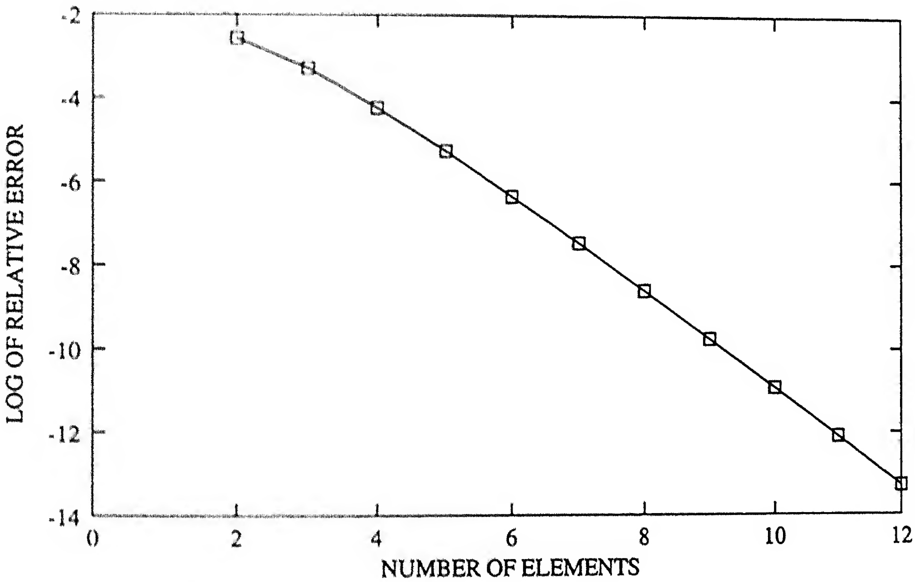

Figure 4.3: Log of relative error vs. N

We now provide the results for the solution of the Schur complement matrix. Table 4.2 shows the number of iterations against the number of elements N. Fig. 4.4 shows the Log of number of iterations on the scale ln (Iterations) against the Log of number of elements $\ln(N)$. Here iterations denotes the minimum number of iterations of the preconditioned conjugate gradient method needed to solve the problem to the maximum accuracy achievable. To check the asymptotic estimate that Iterations $= O\left(N^{3/2} \ln N\right)$,

Table 4.2: Number of iterations against N

N	Iterations
2	205
3	568
4	1053
5	1591
6	2071
7	2522
8	3270
9	3641
10	4637
11	5260
12	5959

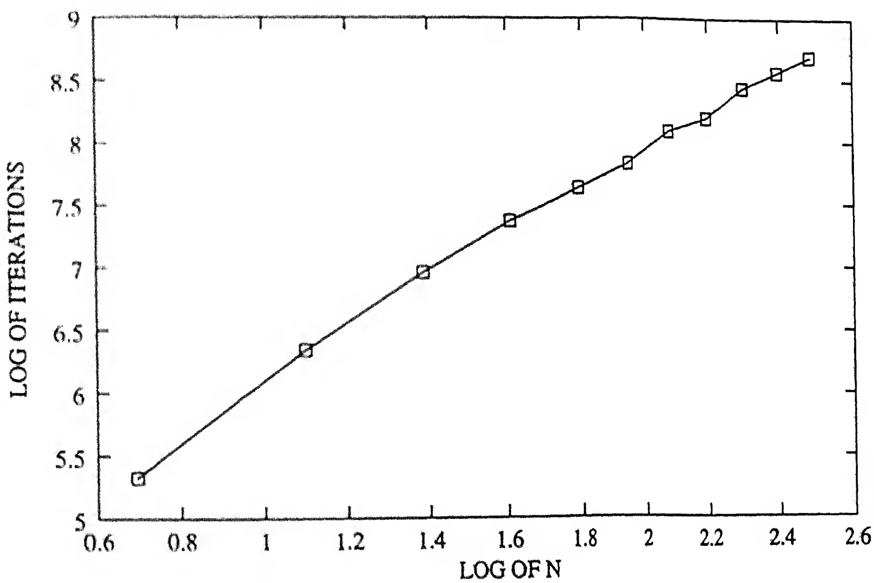

Figure 4.4: Log of iterations vs. log of N

we fit a straight line to the data given points $N = N_0$ to $N = 12$ and compute the slope m. Table 4.3 shows the slope m against N_0. The results confirm the estimate we have obtained.

Table 4.3: Relationship between N_0 and m

N_0	Slope m
2	1.8028
4	1.5575
6	1.5541
8	1.5480

Chapter 5

General Elliptic Problems on Curvilinear Polygons

5.1 Introduction

In this chapter we generalize all the results we have obtained in previous chapters and we seek a solution to an elliptic boundary value problem where the differential operator satisfies the Babuska – Brezzi *inf-sup* condition. We solve the boundary value problem on a curvilinear polygon whose sides are piecewise analytic and we assume the boundary conditions are of mixed Neumann and Dirichlet type as in [7, 8].

We now briefly describe the contents of this chapter. In Section 5.2 we discuss *function spaces* and obtain *differentiability estimates* for the solution in *modified polar coordinates*. In Section 5.3 we obtain a *stability theorem* for an *essentially non-conforming* spectral element representation, viz. spectral element functions which are non-conforming except at the vertices of the element on which they are defined. Finally in Section 5.4 we make some concluding remarks.

5.2 Function Spaces and Differentiability Estimates

Let Ω be a curvilinear polygon with vertices $A_1, A_2, \dots, A_p$ and corresponding sides $\Gamma_1, \Gamma_2, \dots, \Gamma_p$ where Γ_i joins the points A_{i-1} and A_i . We shall assume that the sides $\overline{\Gamma}_i$ are analytic arcs, i.e.

$$\overline{\Gamma}_i = \left\{ (\varphi_i(\xi), \psi_i(\xi)) \,|\, \xi \in \overline{I} = [-1, 1] \right\}$$

with $\varphi_i(\xi)$ and $\psi_i(\xi)$ being analytic functions on $\overline{I}$ and $|\varphi_i'(\xi)|^2 + |\psi_i'(\xi)|^2 \geq \alpha > 0$. By Γ_i we mean the open arc, i.e. the image of $I = (-1, 1)$.

Let the angle subtended at A_j be ω_j.

We shall denote the boundary $\partial\Omega$ of Ω by Γ. Further let $\Gamma = \Gamma^{[0]} \bigcup \Gamma^{[1]}$, $\Gamma^{[0]} = \bigcup_{i\in\mathcal{D}} \overline{\Gamma}_i$, $\Gamma^{[1]} = \bigcup_{i\in\mathcal{N}} \overline{\Gamma}_i$ where $\mathcal{D}$ is a subset of the set $\{i \,|\, i = 1, \dots, p\}$ and $\mathcal{N} = \{i \,|\, i = 1, \dots, p\} \setminus \mathcal{D}$. Let x denote the vector $x = (x_1, x_2)$.

Let $\mathfrak{L}$ be a strongly elliptic operator

$$\mathfrak{L}(u) = -\sum_{r,s=1}^{2} \left(a_{r,s}(x)\, u_{x_s}\right)_{x_r} + \sum_{r=1}^{2} b_r(x)\, u_{x_r} + c(x)\, u \tag{5.1}$$

where $a_{s,r}(x) = a_{r,s}(x)$, $b_r(x)$, $c_r(x)$ are analytic functions on $\overline{\Omega}$ and for any $(\xi_1, \xi_2) \in \mathbb{R}$ and any $x \in \overline{\Omega}$

$$\sum_{r,s=1}^{2} a_{r,s}\xi_r\xi_s \geq \mu_0 \left(\xi_1^2 + \xi_2^2\right) \tag{5.2}$$

with $\mu_0 > 0$. Moreover let the bilinear form induced by the operator $\mathfrak{L}$ satisfy the inf-sup conditions.

In this chapter we shall consider the boundary value problem

$$\begin{aligned} \mathfrak{L}u &= f \quad \text{on } \Omega, \\ u &= g^{[0]} \quad \text{on } \Gamma^{[0]}, \quad \text{and} \\ \left(\frac{\partial u}{\partial N}\right)_A &= g^{[1]} \quad \text{on } \Gamma^{[1]}, \end{aligned} \tag{5.3}$$

where $\left(\frac{\partial u}{\partial N}\right)_A$ denotes the usual conormal derivative which we shall now define. Let A denote the 2×2 matrix whose entries are given by

$$A_{r,s}(x) = a_{r,s}(x)$$

for $r, s = 1, 2$. Let $N = (N_1, N_2)$ denote the outward normal to the curve Γ_i for $i \in \mathcal{N}$. Then $\left(\frac{\partial u}{\partial N}\right)_A$ is defined as follows

$$\left(\frac{\partial u}{\partial N}\right)_A (x) = \sum_{r,s=1}^{2} N_r a_{r,s} \frac{\partial u}{\partial x_s} \tag{5.4}$$

We shall assume that the given data f is analytic on $\overline{\Omega}$ and g^l is analytic on every closed arc $\overline{\Gamma}_i$ and $g^{[0]}$ is continuous on $\Gamma^{[0]}$.

We need to state our regularity estimates in terms of local variables which are defined on a *geometrical mesh* imposed on Ω as in Section 5 of [8]. We first divide Ω into subdomains. Thus we divide Ω into p subdomains $S^1, \ldots, S^p$, where S^i denotes a domain which contains the vertex A^i and no other, and on each S^i we define a geometrical mesh. Let $\mathfrak{S}^k = \{\Omega^k_{i,j},\ j = 1, \ldots, J_k,\ i = 1, \ldots, I_{k,j}\}$ be a partition of $\mathfrak{S}^k$ and let $\mathfrak{S} = \bigcup_{k=1}^{p} \mathfrak{S}^k$.

We now put some restrictions on $\mathfrak{S}$. Let (r_k, θ_k) denote polar coordinates with center at A_k. Let $\tau_k = \ln r_k$. We choose ρ so that the curvilinear sector Ω^k with sides Γ_k and Γ_{k+1}, center at A_k and radius ρ satisfies

$$\Omega^k \subseteq \bigcup_{\Omega^k_{i,j} \in \mathfrak{S}_k} \overline{\Omega}^k_{i,j}.$$

Ω^k may be represented as

$$\Omega^k = \{(x, y) \in \Omega \,|\, 0 < r_k < \rho\}. \tag{5.5}$$

The geometrical mesh we have imposed on Ω is as shown in Fig. 5.1.

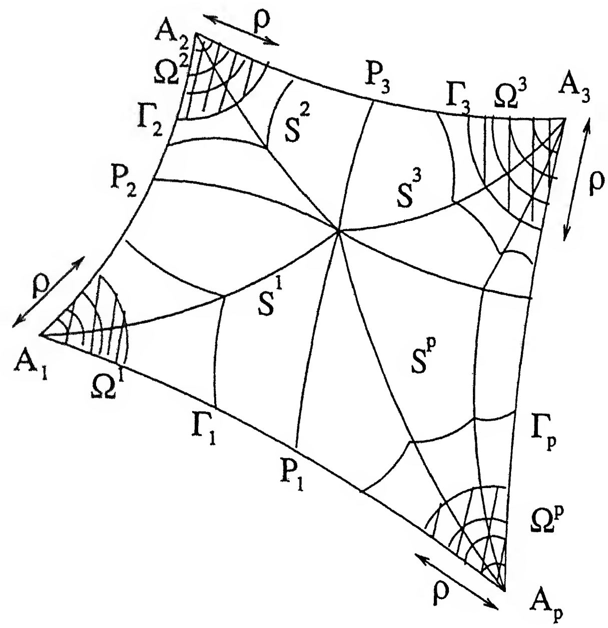

Figure 5.1: Mesh on whole curvilinear domain

Let $\gamma^k_{i,j,l}$, $1 \le l \le 4$ be the side of the quadrilateral $\Omega^k_{i,j} \in \mathfrak{G}$. Then we assume that

$$\text{(5.6a)} \qquad \gamma^k_{i,j,l} : \begin{cases} x = h^k_{i,j}\varphi^k_{i,j,l}(\xi), \\ y = h^k_{i,j}\psi^k_{i,j,l}(\xi), \end{cases} \quad 0 \le \xi \le 1,\ l = 1, 3$$

$$\text{(5.6b)} \qquad \gamma^k_{i,j,l} : \begin{cases} x = h^k_{i,j}\varphi^k_{i,j,l}(\eta), \\ y = h^k_{i,j}\psi^k_{i,j,l}(\eta), \end{cases} \quad 0 \le \eta \le 1,\ l = 2, 4$$

and that for some $C \ge 1$ and $L \ge 1$ independent of i, j, k and l.

$$\text{(5.7)} \qquad \left|\frac{d^t}{ds^t}\varphi^k_{i,j,l}(s)\right|, \left|\frac{d^t}{ds^t}\psi^k_{i,j,l}(s)\right| \le CL^t t!,\ t = 1, 2, \ldots$$

We shall place further restrictions on the geometric mesh we impose on Ω^k later.

Let (r_k, θ_k) be polar coordinates with center at A_k. Then Ω^k is the open set bounded by the curvilinear arcs Γ_k, Γ_{k+1} and a portion of the circle $r_k = \rho$. We subdivide Ω^k into curvilinear rectangles by drawing N circular arcs $r_k = \sigma^k_j = \rho\mu_k^{N+1-j}$, $j = 2, \ldots, N+1$, where $\mu_k < 1$ and $I_k - 1$ analytic curves $C_2, \ldots, C_{I_k}$ whose exact form we shall prescribe

in what follows. We define $\sigma_1^k = 0$. Thus $I_{k,j} = I_k$ for $j \le N$; in fact, we shall let $I_{k,j} = I_k$ for $j \le N+1$.

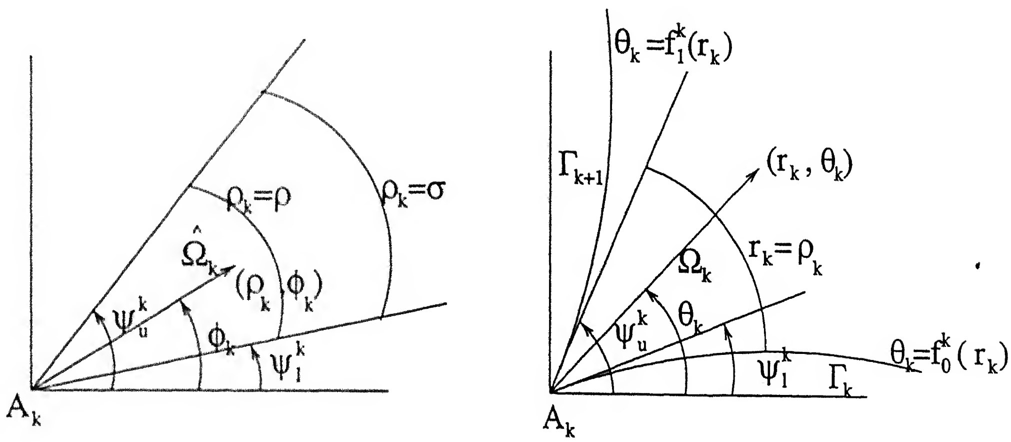

Figure 5.2: Mapping curvilinear corner onto a sector

Moreover $I_{k,j} \le I$ for all k, j where I is a fixed constant. Let

$$\Gamma_{k+j} = \left\{(r_k, \theta_k) \,\middle|\, \theta_k = f_j^k(r_k),\ 0 < r_k < \rho\right\},$$

$j = 0, 1$ in a neighborhood A_k of Ω^k. Then the mapping

$$r_k = \rho_k,\ \theta_k = \frac{1}{\left(\psi_u^k - \psi_l^k\right)}\left[\left(\phi_k - \psi_l^k\right) f_1^k(\rho_k) - \left(\phi_k - \psi_u^k\right) f_0^k(\rho_k)\right], \tag{5.8}$$

where f_j^k is analytic in r_k for $j = 0, 1$, maps locally the cone

$$\left\{(\rho_k, \phi_k) \,\middle|\, 0 < \rho_k < \sigma,\ \psi_l^k < \phi_k < \psi_u^k\right\}$$

onto a set containing Ω^k as in Section 3 of [8]. The functions f_j^k satisfy $f_0^k(0) = \psi_l^k$, $f_1^k(0) = \psi_u^k$ and $f_j^{k\prime}(0) = 0$ for $j = 0, 1$. It is easy to see that the mapping defined in (5.8) has two bounded derivatives in a neighborhood of the origin which contains the closure of the open set

$$\widehat{\Omega}^k = \left\{(\rho_k, \phi_k) \,\middle|\, 0 < \rho_k < \rho,\ \psi_l^k < \phi_k < \psi_u^k\right\}.$$

We choose the I_{k-1} curves $C_2, \dots, C_{I_k}$ as:

$$C_i \left| \phi_k \left(r_k, \theta_k\right) = \psi_i^k \right.$$

for $i = 2, \dots, I_k$. Here

$$\psi_l^k = \psi_1^k < \psi_2^k < \ldots < \psi_{I_k+1}^k = \psi_u^k.$$

Let $\Delta\psi_i^k = \psi_{i+1}^k - \psi_i^k$. Then we choose $\left\{\psi_i^k\right\}_{i,k}$ so that

$$\max_{i,k} \left(\Delta\psi_i^k\right) < \lambda \left(\min_{i,k} \left(\Delta\psi_i^k\right)\right) \tag{5.9}$$

for some constant λ. We need another set of local variables (τ_k, θ_k) in a neighborhood of Ω^k where $\tau_k = \ln r_k$. In addition we need one final set of local variables (ν_k, ϕ_k) in the cone

$$\left\{\left(\rho_k, \phi_k\right) \left| 0 \le \rho_k \le \rho,\ \psi_l^k \le \phi_k \le \psi_u^k \right.\right\},$$

where $\nu_k = \ln \rho_k$.

Let $S_\mu^k = \left\{\left(r_k, \theta_k\right) \left| 0 \le r_k \le \mu \right.\right\} \cap \Omega$. Then the image $\widehat{S}_\mu^k$ in (ν_k, ϕ_k) variables of S_μ^k is given by

$$\widehat{S}_\mu^k = \left\{\left(\nu_k, \phi_k\right) \left| -\infty \le \nu_k \le \ln \mu,\ \psi_i^k \le \phi^k \le \psi_u^k \right.\right\}.$$

Now the relationship between the variables (τ_k, θ_k) and (ν_k, ϕ_k) is given by $(\tau_k, \theta_k) = M^k\left(\nu_k, \phi_k\right)$, viz.

$$\begin{aligned} \tau_k &= \nu_k \\ \theta_k &= \frac{1}{\left(\psi_u^k - \psi_l^k\right)} \left[\left(\phi_k - \psi_l^k\right) f_1^k\left(e^{\nu_k}\right) - \left(\phi_k - \psi_u^k\right) f_0^k\left(e^{\nu_k}\right)\right]. \end{aligned} \tag{5.10}$$

Hence it is easy to see that $J_{M^k}\left(\nu_k, \phi_k\right)$, the Jacobian of the above transformation, satisfies $C_1 \le \left|J_{M^k}\left(\nu_k, \phi_k\right)\right| \le C_2$ for all $(\nu_k, \phi_k) \in \widehat{S}_\mu^k$, for all $0 < \mu \le \rho$.

We should mention here that it is not necessary to choose the system of curves we have chosen to impose a geometric mesh on S_μ^k. However it is necessary to choose the curve $r_k = \rho$ as the boundary of Ω^k and no other, as will become apparent in what follows. Any other additional set of analytic curves which imposes a geometrical mesh on S_μ^k would do equally well. However the set of curves we have chosen is, in some sense, the most natural as the image $\widehat{\Omega}_{i,j}^k$ of a curvilinear rectangle $\Omega_{i,j}^k$ for $j \geq 2$ in (ν_k, ϕ_k) variables is given by a rectangle with straight lines for sides and for $j = 1$ is a semi infinite strip with straight lines for sides.

We now state the differentiability estimates for the solution u of (5.3) which will be needed in this chapter.

Let $U_{i,j}^k(\nu_k, \phi_k) = u(\nu_k, \phi_k)$ for $\nu_k, \phi_k \in \widehat{\Omega}_{i,j}^k$ for $j \leq N$ and $a_k = u(A_k)$. Now there is an analytic mapping $M_{i,j}^k \,|S \to \Omega_{i,j}^k$ for $j > N$ given by $M_{i,j}^k(\xi, \eta) = \left(X_{i,j}^k(\xi, \eta), Y_{i,j}^k(\xi, \eta)\right)$. Here S is the unit square. Let $U_{i,j}^k(\xi, \eta) = u\left(X_{i,j}^k(\xi, \eta), Y_{i,j}^k(\xi, \eta)\right)$. Then we can show as in previous chapters that

(5.11a) $$\left\|U_{i,j}^k(\nu_k, \phi_k) - a_k\right\|_{m,S}^2 \leq \left(Cm!d^m \mu_k^{(1-\beta_k)(N-j+2)}\right)^2$$

for$1 < j \leq N$, $k = 1, \ldots, p$, $1 \leq i \leq I_k$ and

(5.11b) $$\left\|U_{i,j}^k(\xi, \eta)\right\|_{m,S}^2 \leq (Cm!d^m)^2$$

for $N < j \leq J_k$, $1 \leq i \leq I_k$, $1 \leq k \leq p$.

With this we have obtained all the differentiability estimates we shall need to use in this chapter.

5.3 Stability Estimates

Let

(5.12) $$\mathfrak{L}u = -\sum_{r,s=1}^{2} \left(a_{r,s}(x)\, u_{x_s}\right)_{x_r} + \sum_{r=1}^{2} b_r(x)\, u_{x_r} + c(x)\, u$$

be a strongly elliptic operator which satisfies the inf-sup condition.

Hence there exists a positive constant $\mu_0 > 0$ such that

$$\sum_{r,s=1}^{2} a_{r,s}(x)\,\xi_r\xi_s \geq \mu_0\left(\xi_1^2+\xi_2^2\right),$$

for all $x \in \overline{\Omega}$.

Let $H = H_0^1(\Omega)$ where $w \in H_0^1(\Omega)$ if $w \in H^1(\Omega)$ and $\operatorname{trace}(w)|_{\Gamma[0]} = 0$. Consider the bilinear form $B(u,v)$ defined on $H \times H$ as follows

$$B(u,v) = \int_\Omega \left(\sum_{r,s=1}^{2} a_{r,s}(x)\,u_{x_s}v_{x_r} + \sum_{r=1}^{2} b_r(x)\,u_{x_r}v + cuv\right)dx. \tag{5.13}$$

Then $B(u,v)$ is a continuous mapping from $H \times H \to \mathbb{R}$ and there exists a constant C_1 such that

$$|B(u,v)| \leq C_1 \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \tag{5.14}$$

for all $u, v \in H_0^1(\Omega)$.

Moreover

$$\inf_{0\neq u\in H}\ \sup_{0\neq v\in H} \frac{B(u,v)}{\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}} \geq C_2 > 0, \text{ and} \tag{5.15a}$$

$$\sup_{u\in H} B(u,v) > 0 \text{ for every } 0 \neq v \in H. \tag{5.15b}$$

Then for every continuous linear functional $F(v)$ defined on $H_0^1(\Omega)$ there exists unique $u_0 \in H_0^1(\Omega)$ such that $B(u_0,v) = F(v)$ for all $v \in H_0^1(\Omega)$.

Moreover, the *a'priori* estimate

$$\|u_0\|_{H_0^1(\Omega)} \leq \frac{1}{C_2} \sup_{0\neq v\in H_0^1(\Omega)} \frac{|F(v)|}{\|v\|_{H^1(\Omega)}} \tag{5.16}$$

holds.

Now consider the following mixed boundary value problem

$$\mathfrak{L}u = f \text{ in } \Omega, \tag{5.17a}$$

$$\overline{\gamma}_0 u = u|_{\Gamma^{[0]}} = g^{[0]}, \quad \text{and} \tag{5.17b}$$

$$\overline{\gamma}_1 u = \left(\frac{\partial u}{\partial N}\right)_A\Big|_{\Gamma^{[1]}} = g^{[1]}. \tag{5.17c}$$

Here the conormal derivative $\overline{\gamma}_1 u$ is defined as follows.

Let $\Gamma_i \subseteq \Gamma^{[1]}$ and let T and N denote the unit tangent vector and unit outward normal at a point P on Γ_i which we traverse in the clockwise direction. Let $T = (T_1, T_2)^t$ and $N = (N_1, N_2)^t$. Then

$$\overline{\gamma}_l u|_{\Gamma_i} = \left(\frac{\partial u}{\partial N}\right)_A\Big|_{\Gamma_i} = \sum_{r,s=1}^{2} N_r a_{r,s} \frac{\partial u}{\partial x_s} = N^t A \nabla_x u. \tag{5.18a}$$

In the same way we define the cotangential derivative

$$\left(\frac{\partial u}{\partial T}\right)_A\Big|_{\Gamma_i} = \textstyle\sum_{r,s=1}^{2} T_r a_{r,s} \frac{\partial u}{\partial x_s} = T^t A \nabla_x u. \tag{5.18b}$$

With this introductory background we can now proceed to develop the results we shall need to prove the stability theorem, on which our numerical scheme will be based.

Let

$$O^{p+1} = \left\{\Omega^k_{i,j},\ 1 \le k \le p,\ N < j \le J_k,\ 1 \le i \le I_{k,j}\right\}.$$

We shall relabel the elements of O^{p+1} and write

$$O^{p+1} = \left\{\Omega^{p+1}_l, 1 \le l \le L\right\}.$$

Consider some $\Omega^{p+1}_l \in O^{p+1}$, as shown in Figure 5.3. Then Ω^{p+1}_l is a curvilinear quadrilateral whose sides are analytic arcs and the boundary $\partial\Omega^{p+1}_l$ is traversed in the clockwise direction.

Let γ be a smooth curve and let N and T denote the unit outward normal and

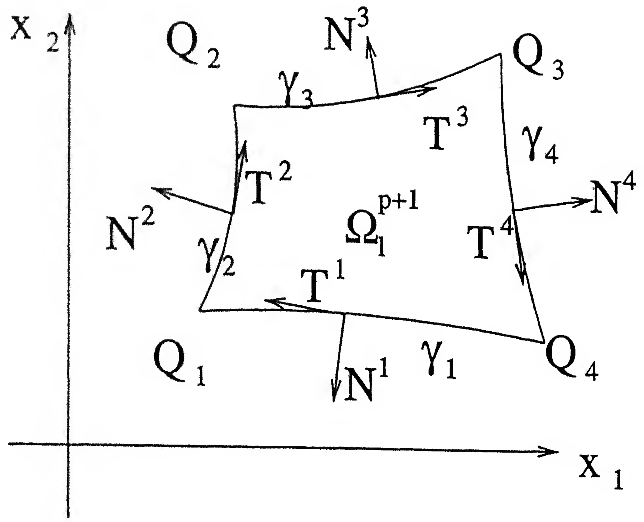

Figure 5.3: N and T vectors on analytic arcs of curvilinear domain

tangent vectors to γ at a point P on γ. Let s be arc length measured from a point on the curve in the clockwise direction. Then the second fundamental form is given by

$$\mathfrak{B}\left(\xi,\eta\right)=-\frac{\partial N}{\partial s}\cdot T\xi\eta=\frac{\partial T}{\partial s}\cdot N\xi\eta=\kappa\xi\eta, \tag{5.19}$$

where $\kappa=\pm\frac{dT}{ds}$ is the curvature of γ at P. Clearly trace $(\mathfrak{B})=\kappa$.

Now we need to use Theorem 3.1.1.2 of [23]. Let v be a smooth vector field defined on $\overline{\Omega}_l^{p+1}$ where $v=(v_1,v_2)^t$. Consider the restriction of v to the boundary $\partial\Omega_l^{p+1}$. Now $\partial\Omega_l^{p+1}=\left(\bigcup_{i=1}^4\gamma_i\right)\bigcup\left(\bigcup_{i=1}^4 Q_i\right)$, where γ_i are the sides of $\partial\Omega_l^{p+1}$ with end points deleted and Q_i are the vertices of Ω_l^{p+1}. We shall denote by v_T the projection of v on the tangent vector T to $\partial\Omega_l^{p+1}$ except at the vertices where this cannot be defined. Similarly by v_N we shall denote the component of v in the direction of N. Thus we have

$$\begin{aligned} v_N &= v\cdot N, \quad \text{and} \\ v_T &= v\cdot T \quad . \end{aligned}$$

Lemma 5.1 *Let $u \in H^2\left(\Omega_l^{p+1}\right)$. Then*

$$
\begin{aligned}
(5.20) \qquad & \frac{\mu_0^2}{2} \sum_{r,s=1}^{2} \int_{\Omega_l^{p+1}} \left| \frac{\partial^2 u}{\partial x_r \partial x_s} \right|^2 dx \\
\leq \; & \int_{\Omega_l^{p+1}} |\mathfrak{M} u|^2 \, dx + \sum_{j=1}^{4} \int_{\gamma_j} |\kappa| \left(\left(\frac{\partial u}{\partial N} \right)_A^2 + \left(\frac{\partial u}{\partial T} \right)_A^2 \right) ds \\
+ \; & \frac{512 M^4}{\mu_0^2} \sum_{r=1}^{2} \int_{\Omega_l^{p+1}} \left| \frac{\partial u}{\partial x_r} \right|^2 dx + 2 \sum_{j=1}^{4} \int_{\gamma_j} \left(\frac{\partial u}{\partial T} \right)_A \frac{d}{ds} \left(\frac{\partial u}{\partial N} \right)_A ds \\
+ \; & \sum_{j=1}^{4} \left\{ \left(\frac{\partial u}{\partial N^{j+1}} \right)_A \left(\frac{\partial u}{\partial T^{j+1}} \right)_A - \left(\frac{\partial u}{\partial N^{j}} \right)_A \left(\frac{\partial u}{\partial T^{j}} \right)_A \right\} (Q_j)
\end{aligned}
$$

We shall say that a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary Γ has a piecewise C^2 boundary if $\Gamma = \Gamma_0 \bigcup \Gamma_1$, where

1. Γ_0 has zero measure (for the arc length measure ds)

2. Γ_1 is open in Γ and each point $x \in \Gamma_1$ has a C^2 boundary as defined in 1.2.1.1 of [23]. Then Theorem 3.1.1.2 of [23] may be stated as follows:

Let O be a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary Γ. Assume in addition that Γ is piecewise C^2. Then for all $v \in H^1(\Omega)$ we have

$$
\begin{aligned}
(5.21) \qquad & \int_O |div\,(v)|^2 \, dx - \int_O \sum_{r,s=1}^{2} \frac{\partial v_r}{\partial x_s} \frac{\partial v_s}{\partial x_r} dx \\
= \; & \int_{\Gamma_1} \left\{ \frac{d}{ds} (v_N v_T) - 2 v_T \frac{d}{ds} v_N \right\} ds - \int_{\Gamma_1} \left\{ (\mathrm{trace}\,(\mathfrak{B}))\, v_N^2 + \mathfrak{B}(v_T, v_T) \right\} ds
\end{aligned}
$$

To apply (5.21) we define the vector field

$$v = A \nabla_x u$$

where A is the matrix

$$(A)_{r,s} = a_{r,s}.$$

We then observe that

$$\text{(5.22a)} \qquad \mathfrak{M}u = \sum_{r,s=1}^{2} \frac{\partial}{\partial x_r}\left(a_{r,s}\frac{\partial u}{\partial x_s}\right) = div\,(v)\,,$$

$$\text{(5.22b)} \qquad \left(\frac{\partial u}{\partial N}\right)_A = \sum_{r,s=1}^{2} N_r a_{r,s}\frac{\partial u}{\partial x_s} = (\overline{\gamma}_0 v)\cdot N, \quad \text{and}$$

$$\text{(5.22c)} \qquad \left(\frac{\partial u}{\partial T}\right)_A = \sum_{r,s=1}^{2} T_r a_{r,s}\frac{\partial u}{\partial x_s} = (\overline{\gamma}_0 v)\cdot T.$$

Hence (5.21) takes the form

$$\text{(5.23)} \qquad \int_{\Omega_l^{p+1}} |\mathfrak{M}u|^2\, dx - \sum_{r,s=1}^{2}\int_{\Omega_l^{p+1}} \frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r}dx$$
$$= \sum_{j=1}^{4}\int_{\gamma_j}\frac{d}{ds}(v_N v_T)\,ds - \sum_{j=1}^{4} 2\int_{\gamma_j}\left(\frac{\partial u}{\partial T}\right)_A \frac{d}{ds}\left(\frac{\partial u}{\partial N}\right)_A ds$$
$$- \sum_{j=1}^{4}\int_{\gamma_j}\kappa\left(\left(\frac{\partial u}{\partial N}\right)_A^2 + \left(\frac{\partial u}{\partial T}\right)_A^2\right) ds\,.$$

Now by Lemma 3.1.3.4 of [23] the following inequality holds for all $u \in H^2(\Omega)$:

$$\mu_0^2 \sum_{r,s=1}^{2}\left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 \le \sum_{r,s,k,l=1}^{2} a_{r,k}a_{s,l}\frac{\partial^2 u}{\partial x_s \partial x_k}\frac{\partial^2 u}{\partial x_r \partial x_l}$$

a.e. in Ω.

Thus it follows that

$$\mu_0^2 \sum_{r,s=1}^{2}\left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 \le \sum_{r,s=1}^{2}\frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r} + 2\sum_{r,s,k,l=1}^{2}\left|a_{r,k}\frac{\partial^2 u}{\partial x_s \partial x_k}\frac{\partial a_{s,l}}{\partial x_r}\frac{\partial u}{\partial x_l}\right|$$

a.e. in Ω. Integrating we have

$$\mu_0^2 \sum_{r,s=1}^{2}\int\left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 dx \le \sum_{r,s=1}^{2}\int\frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r}dx + 32M^2\int_\Omega \sum_{r=1}^{2}\left|\frac{\partial u}{\partial x_r}\right|\sum_{r,s=1}^{2}\left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right| dx$$

where M is a common bound for all the C^1 norms of all the $a_{r,s}$.

Hence

$$(5.24)\quad \frac{\mu_0^2}{2}\sum_{r,s=1}^{2}\int\left|\frac{\partial^2 u}{\partial x_r \partial x_s}\right|^2 dx \le \sum_{r,s=1}^{2}\int \frac{\partial v_r}{\partial x_s}\frac{\partial v_s}{\partial x_r}dx + \frac{2\,(256)\,M^4}{\mu_0^2}\sum_{r=1}^{2}\int\left|\frac{\partial u}{\partial x_r}\right|^2 dx.$$

Next

$$(5.25)\quad \sum_{j=1}^{4}\int_{\gamma_j}\frac{d}{ds}\left(v_N v_T\right)ds$$

$$= \sum_{j=1}^{4}\left\{-\left(\frac{\partial u}{\partial N^{j+1}}\right)_A\left(\frac{\partial u}{\partial T^{j+1}}\right)_A + \left(\frac{\partial u}{\partial N^{j}}\right)_A\left(\frac{\partial u}{\partial T^{j}}\right)_A\right\}(Q_j)$$

Then combining (5.23 - 5.25) we obtain the result.□

In a neighborhood of the vertex A_k we move to polar coordinates. We take a curvilinear rectangle $\Omega_{i,j}^k$ which comprises part of the sectoral neighborhood Ω^k of the vertex A_k and consider it's image $\widetilde{\Omega}_{i,j}^k$ in (τ_k, θ_k) variables as shown in Figure 5.4.

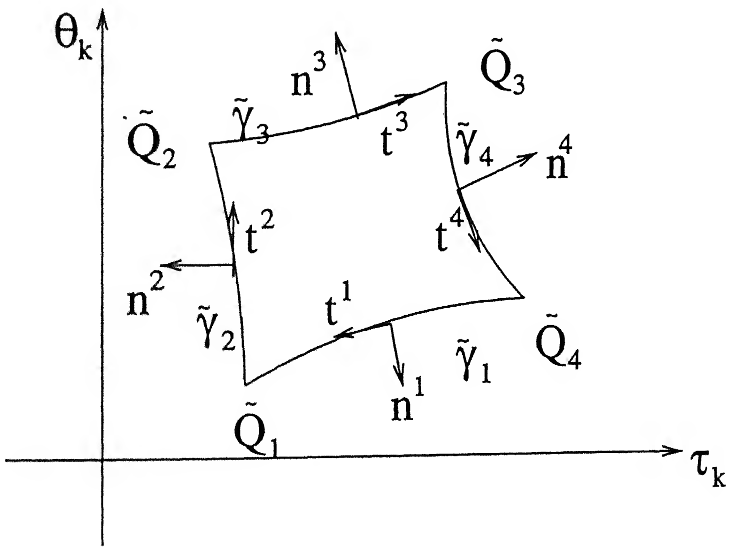

Figure 5.4: n and t vectors on analytic arcs of curvilinear domain in (τ_k, θ_k) coordinates

As in Chapter 2 we write the differential operator $\mathfrak{M}$ in modified polar coordinates, where

$$\mathfrak{M}u = \sum_{r,s=1}^{2}\frac{\partial}{\partial x_r}\left(a_{r,s}\frac{\partial u}{\partial x_s}\right).$$

Now

$$\begin{aligned} x_1 &= x_1^k + e^{\tau_k}\cos\theta_k, \text{ and} \\ x_2 &= x_2^k + e^{\tau_k}\sin\theta_k \quad . \end{aligned}$$

Here $A_k = \left(x_1^k, x_2^k\right)$.

We would like to obtain an estimate for

$$\int_{\Omega_{i,j}^k} r_k^2 \left|\mathfrak{M}u\right|^2 dx = \int_{\tilde{\Omega}_{i,j}^k} \left|\widetilde{\mathfrak{M}}^k u\right|^2 d\tau_k d\theta_k.$$

Let us define the new differential operator

$$\widetilde{\mathfrak{M}}^k u = e^{2\tau_k} \sum_{r,s=1}^{2} \frac{\partial}{\partial x_r}\left(a_{r,s}\frac{\partial u}{\partial x_s}\right) = \sum_{r,s=1}^{2} \frac{\partial}{\partial y_r}\left(\tilde{a}_{r,s}\frac{\partial u}{\partial y_s}\right) \tag{5.26}$$

Here $y_1 = \tau_k$ and $y_2 = \theta_k$.

Let O^k denote the matrix

$$O^k = \begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix} \tag{5.27a}$$

and $\widetilde{A}^k$ denote the matrix

$$\widetilde{A}^k = \begin{bmatrix} \widetilde{a}_{1,1}^k & \widetilde{a}_{1,2}^k \\ \widetilde{a}_{2,1}^k & \widetilde{a}_{2,2}^k \end{bmatrix}.$$

Then it can be easily shown that

$$\widetilde{A}^k = \left(O^k\right)^t A O^k. \tag{5.27b}$$

Hence, since O^k is an orthogonal matrix, we have that

$$\sum_{r,s=1}^{2} \widetilde{a}_{r,s}^k \eta_r \eta_s \geq \mu_0\left(\eta_1^2 + \eta_2^2\right). \tag{5.28}$$

Moreover the following relations hold

(5.29a) $$\left(\widetilde{a}_{1,1}^{k}\right)_{\theta_k} = 2\widetilde{a}_{1,2}^{k} + O\left(e^{\tau_k}\right),$$

(5.29b) $$\left(\widetilde{a}_{1,2}^{k}\right)_{\theta_k} = \widetilde{a}_{2,2}^{k} - \widetilde{a}_{1,1}^{k} + O\left(e^{\tau_k}\right),$$

(5.29c) $$\left(\widetilde{a}_{2,2}^{k}\right)_{\theta_k} = -2\widetilde{a}_{1,2}^{k} + O\left(e^{\tau_k}\right) \text{ and}$$

(5.29d) $$\left(\widetilde{a}_{1,1}^{k}\right)_{\tau_k}, \left(\widetilde{a}_{1,2}^{k}\right)_{\tau_k} \text{ and } \left(\widetilde{a}_{2,2}^{k}\right)_{\tau_k} = O\left(e^{\tau_k}\right)$$

as $\tau_k \to -\infty$.

Next let γ be a curve given by

$$\begin{aligned} x_1 &= x_1(s) \\ x_2 &= x_2(s) \end{aligned}$$

where s is arc length along the curve γ. Then the curvature κ at a point P on the curve is given by

$$\kappa = \frac{dx_1}{ds}\frac{d^2x_2}{ds^2} - \frac{dx_2}{ds}\frac{d^2x_1}{ds^2}.$$

Let $\widetilde{\gamma}$ be the image of the curve in (y_1, y_2) coordinate given by

$$\begin{aligned} y_1 &= y_1(\sigma), \\ y_2 &= y_2(\sigma) \end{aligned}$$

where σ is arc length along the curve $\widetilde{\gamma}$. Then it is easy to verify that

(5.30) $$\frac{ds}{d\sigma} = e^{y_1}.$$

Now we can show that the curvature $\widetilde{\kappa}$ of the curve $\widetilde{\gamma}$ is given by

$$\widetilde{\kappa} = \kappa e^{y_1} + \frac{dy_2}{d\sigma}.$$

Hence

(5.31) $$|\widetilde{\kappa}| < |\kappa| e^{\tau_k} + 1 \leq K$$

where K is a uniform constant for all the curves $\widetilde{\gamma}_s \subseteq \widetilde{\Omega}^k$.

We shall denote by t and n the unit tangent and outward normal vector at a point P on $\widetilde{\gamma}$, the boundary of $\widetilde{\Omega}^k_{i,j}$ except at it's vertices where these are not defined.

Lemma 5.2 *Let* $u(y) \in H^2\left(\widetilde{\Omega}^k_{i,j}\right)$. *Then*

$$\begin{aligned}
(5.32) \qquad & \frac{\mu_0^2}{2}\sum_{r,s=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\left|\frac{\partial^2 u}{\partial y_r \partial y_s}\right| dy \\
\leq & \int_{\widetilde{\Omega}^k_{i,j}}\left|\widetilde{\mathfrak{M}}^k u\right|^2 dy + 2\sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\frac{d}{d\sigma}\left(\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right)d\sigma \\
+ & \sum_{j=1}^{4}\left\{\left(\frac{\partial u}{\partial t^{j+1}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^{j+1}}\right)_{\widetilde{A}^k} - \left(\frac{\partial u}{\partial t^{j}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^{j}}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{Q}_j\right) \\
+ & \sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}|\widetilde{\kappa}|\left(\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}^2 + \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^2\right)d\sigma + \frac{512}{\mu_0^2}M^4\sum_{r=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\left|\frac{\partial u}{\partial y_r}\right|^2 dy.
\end{aligned}$$

Now once more we use Theorem 3.1.1.2 of [23]. Clearly $\widetilde{\Omega}^k_{i,j}$ for $j \geq 2$ is a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary $\widetilde{\Gamma}$ that is a piecewise C^2. Thus $\widetilde{\Gamma} = \left(\bigcup_{i=1}^4 \widetilde{\gamma}_i\right) \cup \left(\bigcup_{i=1}^4 \widetilde{Q}_i\right)$ where $\widetilde{\gamma}_i$ are the sides of the open rectangle $\widetilde{\Omega}^k_{i,j}$ with the end points removed and $\widetilde{Q}_i$ are it's vertices.

Now

$$\int_{\Omega^k_{i,j}} r_k^2 |\mathfrak{M}u|^2 dx = \int_{\widetilde{\Omega}^k_{i,j}} e^{4\tau_k}|\mathfrak{M}u|^2 d\tau_k d\theta_k = \int_{\widetilde{\Omega}^k_{i,j}}\left|\widetilde{\mathfrak{M}}^k u\right|^2 dy.$$

Here

$$\widetilde{\mathfrak{M}}^k u = \sum_{r,s=1}^{2}\frac{\partial}{\partial y_r}\left(\widetilde{a}^k_{r,s}\frac{\partial u}{\partial y_s}\right)$$

as defined in (5.26).

Then for all $w \in H^1\left(\widetilde{\Omega}^k_{i,j}\right)$ we have

$$\begin{aligned}
(5.33) \qquad & \int_{\widetilde{\Omega}^k_{i,j}}|div(w)|^2 dy - \sum_{r,s=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\frac{\partial w_r}{\partial y_s}\frac{\partial w_s}{\partial y_r}dy \\
= & \sum_{j=1}^{4}\left\{\int_{\widetilde{\gamma}_j}\frac{d}{d\sigma}(w_n w_t) - 2w_t\frac{d}{d\sigma}w_n\right\}d\sigma - \sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}\widetilde{\kappa}\left(w_n^2 + w_t^2\right)d\sigma.
\end{aligned}$$

Here w_n and w_t are the projections of w on the normal and tangent vectors n and t respectively. We define

$$w = \widetilde{A}^k \nabla_y u.$$

Then

$$\text{(5.34a)} \qquad \widetilde{\mathfrak{M}}^k u = \sum_{r,s=1}^{2} \frac{\partial}{\partial y_r}\left(\widetilde{a}^k_{r,s}\frac{\partial u}{\partial y_s}\right) = div\,(w),$$

$$\text{(5.34b)} \qquad \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} = \sum_{r,s=1}^{2} n_r \widetilde{a}^k_{r,s}\frac{\partial u}{\partial y_s} = w_n, \quad \text{and}$$

$$\text{(5.34c)} \qquad \left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k} = \sum_{r,s=1}^{2} t_r \widetilde{a}^k_{r,s}\frac{\partial u}{\partial y_s} = w_t.$$

So (5.33) takes the form

$$\text{(5.35)} \quad \int_{\widetilde{\Omega}^k_{i,j}} \left|\widetilde{\mathfrak{M}}^k u\right|^2 dy - \sum_{r,s=1}^{2} \frac{\partial w_r}{\partial y_s}\frac{\partial w_s}{\partial y_r} dy$$
$$= -2\sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\frac{d}{d\sigma}\left(\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right)d\sigma - \sum_{j=1}^{4}\int \widetilde{\kappa}\left(\left(\frac{\partial u}{\partial t}\right)^2_{\widetilde{A}^k} + \left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}\right)d\sigma$$
$$- \sum_{j=1}^{4}\left\{\left(\frac{\partial u}{\partial t^{j+1}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^{j+1}}\right)_{\widetilde{A}^k} - \left(\frac{\partial u}{\partial t^{j}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^{j}}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{Q}_j\right).$$

Now using Lemma 3.1.3.4 of [23] we obtain

$$\mu_0^2 \sum_{r,s=1}^{2}\left|\frac{\partial^2 u}{\partial y_r \partial y_s}\right|^2 \leq \sum_{i,j=1}^{2}\frac{\partial w_r}{\partial y_s}\frac{\partial w_s}{\partial y_r} + 2\sum_{r,s,t,l=1}^{2}\left|\widetilde{a}^k_{r,t}\frac{\partial^2 u}{\partial y_s \partial y_t}\frac{\partial \widetilde{a}^k_{s,l}}{\partial y_r}\frac{\partial u}{\partial y_l}\right|.$$

And by (5.29a - 5.29d) there exists a constant M such that M is a common bound for the C^1 norms of all the $\tilde{a}^k_{i,j}$. Hence

$$\text{(5.36)} \qquad \frac{\mu_0^2}{2}\sum_{r,s=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\left|\frac{\partial^2 u}{\partial y_r \partial y_s}\right|^2 dy$$
$$\leq \sum_{r,s=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\frac{\partial w_r}{\partial y_s}\frac{\partial w_s}{\partial y_r}dy + \frac{2\,(256)}{\mu_0^2}M^4\sum_{r=1}^{2}\int_{\widetilde{\Omega}^k_{i,j}}\left|\frac{\partial u}{\partial y_r}\right|^2 dy.$$

Thus combining (5.33), (5.35) and (5.36) we get the result.□

We now need to write terms such as

$$2\rho^2 \int_{\gamma_j} \left(\frac{\partial u}{\partial T}\right)_A \frac{d}{ds}\left(\frac{\partial u}{\partial N}\right)_A ds$$

in (5.32) where $\gamma_j \subseteq B_\rho^k = \{(x_1, x_2)\,|\rho_k = \rho\}$ in terms of (y_1, y_2) coordinates. Let γ be a smooth curve in $\Omega_\mu^k = \{(x_1, x_2)\,|(x_1, x_2) \in \Omega \text{ and } \rho_k < \mu\}$, where $\rho < \mu$, and let P be a point on γ such that P in polar coordinates has the representation (ρ_k, θ_k) with $\rho_k = \rho$.

Now

$$e^{y_1}\nabla_x u = O^k \nabla_y u \tag{5.37}$$

where O^k is the matrix defined in (5.27a), And

$$\begin{aligned} T &= O^k t, \quad \text{and} \\ N &= O^k n. \end{aligned} \tag{5.38}$$

Hence

$$\begin{aligned} e^{y_1}\left(\frac{\partial u}{\partial T}\right)_A (P) &= t^t \left(O^k\right)^t A O^k \nabla_y u \left(\widetilde{P}\right) \\ &= t^t \widetilde{A}^k \nabla_y u \left(\widetilde{P}\right) \\ &= \left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k} \left(\widetilde{P}\right) \end{aligned} \tag{5.39a}$$

using (5.27a), (5.37) and (5.38). Here $\widetilde{P}$ is the image of the point P in (y_1, y_2) coordinates. Similarly, we have

$$e^{y_1}\left(\frac{\partial u}{\partial N}\right)_A (P) = \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} \left(\widetilde{P}\right). \tag{5.39b}$$

Thus we can conclude that

$$2\rho^2 \int_{\gamma_j} \left(\frac{\partial u}{\partial T}\right)_A \frac{d}{ds}\left(\frac{\partial u}{\partial N}\right)_A ds = 2\int_{\widetilde{\gamma}_j} \left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k} \frac{d}{d\sigma}\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} d\sigma, \text{ and} \tag{5.40a}$$

$$\left\{\rho^2 \left(\frac{\partial u}{\partial T}\right)_A \left(\frac{\partial u}{\partial N}\right)_A\right\}(P) = \left\{\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k} \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{P}\right). \tag{5.40b}$$

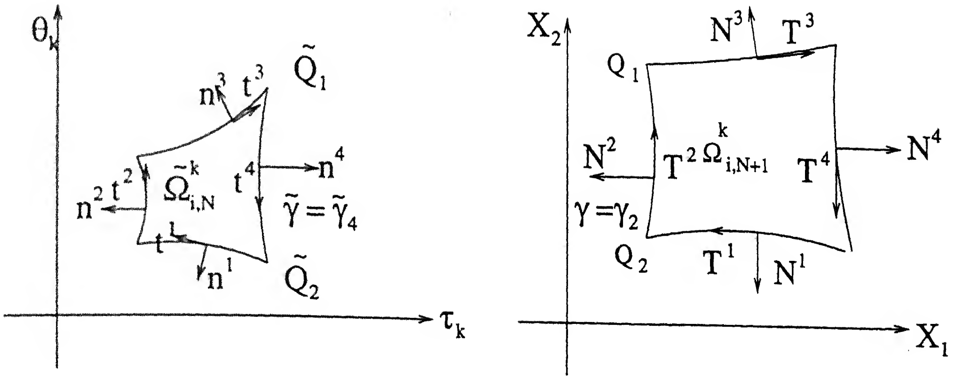

Figure 5.5: Common boundary between $\Omega^k_{i,N+1}$ and $\Omega^k_{i,N}$

Consider the boundary γ common to $\Omega^k_{i,N+1}$ and $\Omega^k_{i,N}$. Then the following relations hold (Fig. 5.5):

$$\text{(5.41a)}\quad \left\{\rho^2\left(\frac{\partial u}{\partial T^3}\right)_A\left(\frac{\partial u}{\partial N^3}\right)_A\right\}(Q_1) = \left\{\left(\frac{\partial u}{\partial t^3}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^3}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{Q}_1\right),$$

$$\text{(5.41b)}\quad \left\{\rho^2\left(\frac{\partial u}{\partial T^2}\right)_A\left(\frac{\partial u}{\partial N^2}\right)_A\right\}(Q_1) = \left\{\left(\frac{\partial u}{\partial t^4}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^4}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{Q}_1\right),$$

$$\text{(5.41c)}\quad \left\{\rho^2\left(\frac{\partial u}{\partial T^2}\right)_A\left(\frac{\partial u}{\partial N^2}\right)_A\right\}(Q_2) = \left\{\left(\frac{\partial u}{\partial t^4}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^4}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{Q}_2\right), \text{ and}$$

$$\text{(5.41d)}\quad \left\{\rho^2\left(\frac{\partial u}{\partial T^1}\right)_A\left(\frac{\partial u}{\partial N^1}\right)_A\right\}(Q_2) = \left\{\left(\frac{\partial u}{\partial t^1}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial n^1}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{Q}_2\right).$$

Now let $\widetilde{\gamma}_l \subseteq \partial\widetilde{\Omega}^k_{i,j}$ for some $j \le N$ and further suppose $\widetilde{\gamma}_l \subseteq \widetilde{\Gamma}_j$ where $j \in \mathcal{D}$. Let n and t be the unit outward normal and tangent vectors, respectively, defined at every point of $\widetilde{\gamma}_l$. Then

$$\text{(5.42)}\qquad \left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}(\sigma) = \widetilde{g}^k(\sigma)\left(\frac{\partial u}{\partial t}\right)(\sigma) + \widetilde{h}^k(\sigma)\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}(\sigma).$$

Here σ is arc length measured from the point $\widetilde{G}$ (Fig. 5.6) where

$$\begin{aligned}\widetilde{g}^k(\sigma) &= t^t\widetilde{A}^k t(\sigma) - \frac{\left(t^t\widetilde{A}^k n(\sigma)\right)^2}{n^t\widetilde{A}^k n(\sigma)}, \text{ and}\\ \widetilde{h}^k(\sigma) &= \frac{t^t\widetilde{A}^k n(\sigma)}{n^t\widetilde{A}^k n(\sigma)}.\end{aligned}$$

Hence

$$\int_{\widetilde{\gamma}_l}\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\frac{d}{d\sigma}\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}d\sigma$$
$$=\int_{\widetilde{\gamma}_l}\widetilde{g}^k(\sigma)\frac{\partial u}{\partial t}\frac{d}{d\sigma}\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}d\sigma+\int_{\widetilde{\gamma}_l}\frac{\widetilde{h}^k(\sigma)}{2}\frac{d}{d\sigma}\left(\left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}\right)d\sigma.$$

And so we can conclude that

$$\int_{\widetilde{\gamma}_l}\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\frac{d}{d\sigma}\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}d\sigma \tag{5.43}$$
$$=\int_{\widetilde{\gamma}_l}\widetilde{g}^k(\sigma)\frac{\partial u}{\partial t}\frac{d}{d\sigma}\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}d\sigma-\frac{1}{2}\int_{\widetilde{\gamma}_l}\frac{d\widetilde{h}^k}{d\sigma}\left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}d\sigma+\frac{\widetilde{h}^k(\sigma)}{2}\left(\frac{\partial u}{\partial n}\right)^2_{\widetilde{A}^k}\Bigg|_{\partial\widetilde{\gamma}_l}.$$

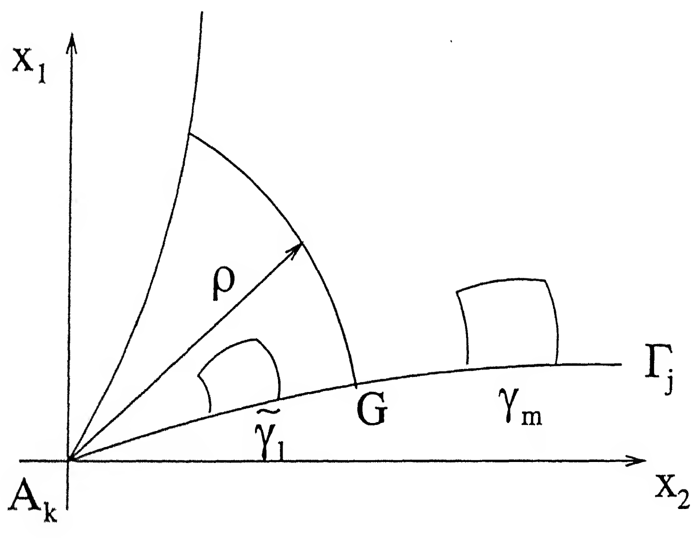

Figure 5.6: Point on a curvilinear boundary

Next let $\gamma_m \subseteq \partial\Omega^k_{i,j}$ for some $j > N$ be such that $\gamma_m \subseteq \Gamma_j$ where $j \in \mathcal{D}$. Let N and T be the unit normal and tangent vectors, respectively, defined at every point of γ_m. Then

$$\left(\frac{\partial u}{\partial T}\right)_A(s)=g(s)\left(\frac{\partial u}{\partial T}\right)(s)+h(s)\left(\frac{\partial u}{\partial N}\right)_A(s). \tag{5.44}$$

where s is arc length measured from the point G as shown in Fig. 5.6.

Here

$$\begin{aligned} g(s) &= T^t A T - \frac{(T^t A N)^2}{N^t A N}, \text{ and} \\ h(s) &= \frac{T^t A N}{N^t A N} \quad . \end{aligned}$$

So we obtain

$$\begin{aligned} (5.45) \quad & \rho^2 \int_{\gamma_m} \left(\frac{\partial u}{\partial T}\right)_A \frac{d}{ds} \left(\frac{\partial u}{\partial N}\right)_A ds \\ = \quad & \rho^2 \int_{\gamma_m} g(s) \frac{\partial u}{\partial T} \frac{d}{ds} \left(\frac{\partial u}{\partial N}\right)_A ds - \frac{\rho^2}{2} \int_{\gamma_m} \frac{dh}{ds} \left(\frac{\partial u}{\partial N}\right)_A^2 ds + \frac{\rho^2 h}{2} \left(\frac{\partial u}{\partial N}\right)_A^2 \Bigg|_{\partial \gamma_m} . \end{aligned}$$

Now by (5.39b) we have that

$$\rho^2 \left(\frac{\partial u}{\partial N}\right)_A^2 (G) = \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^2 \left(\widetilde{G}\right).$$

And moreover by (5.27a) and (5.38)

$$\begin{aligned} (5.46a) \qquad & g(G) &= \widetilde{g}^k \left(\widetilde{G}\right), \text{ and} \\ (5.46b) \qquad & h(G) &= \widetilde{h}^k \left(\widetilde{G}\right). \end{aligned}$$

Recall that

$$\begin{aligned} (5.47) \qquad \mathfrak{L}u &= -\sum_{r,s=1}^{2} \left(a_{r,s}(x)\, u_{x_s}\right)_{x_r} + \sum_{r=1}^{2} b_r(x)\, u_{x_r} + c(x)\, u \\ &= \mathfrak{M}u + \mathfrak{N}u \end{aligned}$$

where

$$\mathfrak{N}u = \sum_{r=1}^{2} b_r(x)\, u_{x_r} + c(x)\, u.$$

Hence

$$\rho^2 \int_{\Omega_l^{p+1}} |\mathfrak{M}u|^2\, dx \leq 2\rho^2 \int_{\Omega_l^{p+1}} |\mathfrak{L}u|^2\, dx + 2\rho^2 \int_{\Omega_l^{p+1}} |\mathfrak{N}u|^2\, dx.$$

Hence using Lemma 5.1 we can conclude that there is a constant C such that

$$
\begin{aligned}
(5.48)\qquad & \frac{\rho^2\mu_0^2}{2}\sum_{r,s=1}^{2}\int_{\Omega_l^{p+1}}\left|\frac{\partial^2 u_l^{p+1}}{\partial x_r \partial x_s}\right|^2 dx \\
& - C\rho^2\left(\sum_{r=1}^{2}\left(\int_{\Omega_l^{p+1}}\left|\frac{\partial u_l^{p+1}}{\partial x_r}\right|^2 dx\right)+\int_{\Omega_l^{p+1}}\left|u_l^{p+1}\right|^2 dx\right) \\
\leq\; & 2\rho^2\int_{\Omega_l^{p+1}}\left|\mathfrak{L}u_l^{p+1}\right|^2 dx + 2\rho^2\sum_{j=1}^{4}\int_{\gamma_j}\left(\frac{\partial u_l^{p+1}}{\partial T}\right)_A \frac{d}{ds}\left(\frac{\partial u_l^{p+1}}{\partial N}\right)_A ds \\
& + \sum_{j=1}^{4}\rho^2\left\{\left(\frac{\partial u_l^{p+1}}{\partial N^{j+1}}\right)_A\left(\frac{\partial u_l^{p+1}}{\partial T^{j+1}}\right)_A - \left(\frac{\partial u_l^{p+1}}{\partial N^{j}}\right)_A\left(\frac{\partial u_l^{p+1}}{\partial T^{j}}\right)_A\right\}(Q_j) \\
& + \rho^2\sum_{j=1}^{4}\int_{\gamma_j}|\kappa|\left(\left(\frac{\partial u_l^{p+1}}{\partial N}\right)_A^2+\left(\frac{\partial u_l^{p+1}}{\partial T}\right)_A^2\right) ds.
\end{aligned}
$$

In the same way we have

$$
\begin{aligned}
\widetilde{\mathfrak{L}}^k u &= e^{2y_1}\left(-\sum_{r,s=1}^{2}\left(a_{r,s}(x)\, u_{x_s}\right)_{x_r}+\sum_{r=1}^{2} b_r(x)\, u_{x_r}+c(x)\, u\right) \\
&= \left(\sum_{r,s=1}^{2}-\left(\widetilde{a}_{r,s}^k(y)\, u_{y_s}\right)_{y_r}\right)+\left(\sum_{r=1}^{2}\widetilde{b}_r^k(y)\, u_{y_r}+\widetilde{c}^k(y)\, u\right) \\
&= \widetilde{\mathfrak{M}}^k u+\widetilde{\mathfrak{N}}^k u.
\end{aligned}
$$

Here

$$
(5.49)\qquad \widetilde{\mathfrak{N}}^k u=\sum_{r=1}^{2}\widetilde{b}_r^k(y)\, u_{y_r}+\widetilde{c}^k(y)\, u
$$

and $y=(y_1,y_2)=(\tau_k,\theta_k)$ for some k.

Moreover the coefficients of $\widetilde{\mathfrak{N}}^k$ satisfy

$$
\begin{aligned}
\widetilde{b}_r^k &= O\left(e^{\tau_k}\right) \text{ for } r=1,2, \text{ and} \\
\widetilde{c}^k &= O\left(e^{2\tau_k}\right)
\end{aligned}
$$

as $\tau_k \to -\infty$.

Once more

$$\int_{\widetilde{\Omega}_{i,j}^k} \left|\widetilde{\mathfrak{M}}^k u\right|^2 dy \leq 2\left(\int_{\widetilde{\Omega}_{i,j}^k} \left|\widetilde{\mathfrak{L}}^k u\right|^2 dy + \int_{\widetilde{\Omega}_{i,j}^k} \left|\widetilde{\mathfrak{N}}^k u\right|^2 dy\right).$$

Using Lemma 5.2 we can conclude that

$$\begin{aligned}
(5.50) \qquad & \frac{\mu_0^2}{2}\sum_{r,s=1}^{2}\int_{\widetilde{\Omega}_{i,j}^k}\left|\frac{\partial^2 u}{\partial y_r \partial y_s}\right|^2 dy - C\left(\sum_{r=1}^{2}\int_{\widetilde{\Omega}_{i,j}^k}\left|\frac{\partial u}{\partial y_r}\right|^2 dy + \int_{\widetilde{\Omega}_{i,j}^k}|u|^2 e^{4y_1}dy\right)\\
\leq\ & 2\int_{\widetilde{\Omega}_{i,j}^k}\left|\widetilde{\mathfrak{L}}^k u\right|^2 dy + 2\sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}\left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}\frac{d}{d\sigma}\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} d\sigma\\
+\ & \sum_{j=1}^{4}\left\{\left(\frac{\partial u}{\partial n^{j+1}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial t^{j+1}}\right)_{\widetilde{A}^k} - \left(\frac{\partial u}{\partial n^{j}}\right)_{\widetilde{A}^k}\left(\frac{\partial u}{\partial t^{j}}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{Q}_j\right)\\
+\ & \sum_{j=1}^{4}\int_{\widetilde{\gamma}_j}|\widetilde{\kappa}|\left(\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^2 + \left(\frac{\partial u}{\partial t}\right)_{\widetilde{A}^k}^2\right)d\sigma.
\end{aligned}$$

We now rewrite (5.50) in (ν_k, ϕ_k) coordinates as:

$$\begin{aligned}
(5.51) \qquad & \beta\sum_{|\alpha|=2}\int_{\widehat{\Omega}_{i,j}^k}\int\left|D_{\nu_k}^{\alpha_1}D_{\phi_k}^{\alpha_2}u_{i,j}^k\right|^2 d\nu_k d\phi_k\\
-\ & C\left(\sum_{|\alpha|=1}\int_{\widehat{\Omega}_{i,j}^k}\int\left|D_{\nu_k}^{\alpha_1}D_{\phi_k}^{\alpha_2}u_{i,j}^k\right|^2 d\nu_k d\phi_k + \int_{\widehat{\Omega}_{i,j}^k}\int\left|u_{i,j}^k\right|^2 e^{4\nu_k}d\nu_k d\phi_k\right)\\
\leq\ & K\int_{\widehat{\Omega}_{i,j}^k}\int\left|\mathfrak{L}^k u_{i,j}^k\right|^2 d\nu_k d\phi_k + 2\sum_{r=1}^{4}\int_{\widetilde{\gamma}_r}\left(\frac{\partial u_{i,j}^k}{\partial t}\right)_{\widetilde{A}^k}\frac{d}{d\sigma}\left(\frac{\partial u_{i,j}^k}{\partial n}\right)_{\widetilde{A}^k} d\sigma\\
+\ & \sum_{r=1}^{4}\left\{\left(\frac{\partial u_{i,j}^k}{\partial n^{r+1}}\right)_{\widetilde{A}^k}\left(\frac{\partial u_{i,j}^k}{\partial t^{r+1}}\right)_{\widetilde{A}^k} - \left(\frac{\partial u_{i,j}^k}{\partial n^{r}}\right)_{\widetilde{A}^k}\left(\frac{\partial u_{i,j}^k}{\partial t^{r}}\right)_{\widetilde{A}^k}\right\}\left(\widetilde{Q}_r\right)\\
+\ & \sum_{r=1}^{4}\int_{\widetilde{\gamma}_r}|\widetilde{\kappa}|\left(\left(\frac{\partial u_{i,j}^k}{\partial n}\right)_{\widetilde{A}^k}^2 + \left(\frac{\partial u_{i,j}^k}{\partial t}\right)_{\widetilde{A}^k}^2\right)d\sigma.
\end{aligned}$$

Here $\widehat{\Omega}_{i,j}^k = \left(\psi_i^k, \psi_{i+1}^k\right) \times \left(\alpha_j^k, \alpha_{j+1}^k\right)$ and β is a positive constant.

In the same way we can rewrite (5.48) in (ξ, η) coordinate as

$$(5.52) \qquad \sum_{|\alpha|=2}\int_S\int\left|D_\xi^{\alpha_1}D_\eta^{\alpha_2}u_l^{p+1}(\xi,\eta)\right|^2 d\xi d\eta - C\left(\sum_{|\alpha|\leq 1}\int_S\int\left|D_\xi^{\alpha_1}D_\eta^{\alpha_2}u_l^{p+1}\right|^2 d\xi d\eta\right)$$

$$
\begin{aligned}
&\leq K\int_S\int\left|\mathfrak{L}_l^{p+1}u_l^{p+1}\right|^2 d\xi d\eta + 2\rho^2\sum_{r=1}^{4}\int\left(\frac{\partial u_l^{p+1}}{\partial T}\right)_A\frac{d}{ds}\left(\frac{\partial u_l^{p+1}}{\partial N}\right)_A ds \\
&+ \sum_{r=1}^{4}\rho^2\left\{\left(\frac{\partial u_l^{p+1}}{\partial N^{r+1}}\right)_A\left(\frac{\partial u_l^{p+1}}{\partial T^{r+1}}\right)_A - \left(\frac{\partial u_l^{p+1}}{\partial N^r}\right)_A\left(\frac{\partial u_l^{p+1}}{\partial T^r}\right)_A\right\}(Q_r) \\
&+ \sum_{r=1}^{4}\int_{\gamma_r}|\kappa|\,\rho^2\left(\left(\frac{\partial u_l^{p+1}}{\partial N}\right)_A^2+\left(\frac{\partial u_l^{p+1}}{\partial T}\right)_A^2\right)ds.
\end{aligned}
$$

Here S is the unit square and $\mathfrak{L}_l^{p+1}$ is the differential operator $\mathfrak{L}$ written in (ξ,η) coordinates.

We now need to obtain estimates for the spectral element functions in the H^1 norm which we do in the following theorem.

Theorem 5.1 *The following estimate holds*

$$
\begin{aligned}
(5.53)\quad &\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left|u_{i,1}^k\right|^2 + \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|u_{i,j}^k(\nu_k,\phi_k)\right\|_{1,\widehat{\Omega}_{i,j}^k}^2 + \sum_{l=1}^{L}\left\|u_l^{p+1}(\xi,\eta)\right\|_{1,S}^2 \\
&\leq CN^4\left\{\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\mathfrak{L}^k u_{i,j}^k(\nu_k,\phi_k)\right\|_{0,\widehat{\Omega}_{i,j}^k}^2\right. \\
&+ \sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k}\left(\|[u]\|_{0,\widehat{\gamma}_s}^2+\|[u_{\nu_k}]\|_{0,\widehat{\gamma}_s}^2+\|[u_{\phi_k}]\|_{0,\widehat{\gamma}_s}^2\right) \\
&+ \sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty}\left(\|u\|_{0,\widehat{\gamma}_s}^2+\|u_{\nu_k}\|_{0,\widehat{\gamma}_s}^2\right) \\
&+ \sum_{k=1}^{p}\sum_{\gamma_s\subseteq B_\rho^k}\left(\|[u]\|_{0,\widehat{\gamma}_s}^2+\|[u_{\nu_k}]\|_{0,\widehat{\gamma}_s}^2+\|[u_{\phi_k}]\|_{0,\widehat{\gamma}_s}^2\right) \\
&+ \sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l}\left\|\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right\|_{0,\widehat{\gamma}_s}^2 \\
&+ \sum_{l=1}^{L}\int_{\Omega_l^{p+1}}\int\left|\mathfrak{L}u_l^{p+1}(\xi,\eta)\right|^2 d\xi d\eta \\
&+ \sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\|[u]\|_{0,\gamma_s}^2+\|[u_{x_1}]\|_{0,\gamma_s}^2+\|[u_{x_2}]\|_{0,\gamma_s}^2\right) \\
&+ \left.\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left(\|u\|_{0,\gamma_s}^2+\left\|\frac{\partial u}{\partial T}\right\|_{0,\gamma_s}^2\right)+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left\|\left(\frac{\partial u}{\partial N}\right)_A\right\|_{0,\gamma_s}^2\right\}.
\end{aligned}
$$

To prove the estimate (5.53) we shall use (5.16). To do so we have to define a

corrected version of the spectral element functions so that it is conforming.

Let $\left\{\left\{u_{i,j}^k\left(\nu_k,\phi_k\right)\right\}_{i,j\leq N,k},\left\{u_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j>N,k}\right\}_k$ be a set of spectral element functions $\in S_V^N$. Here S_V^N is the set of spectral element functions such that $u_{i,1}^k = b_k$, a constant for all i, $u_{i,j}^k$ is a polynomial of degree N in each variable for $j \geq 2$ and the spectral element functions are continuous at the vertices of the domains on which they are defined. Then there is a set of spectral element functions, as in Chapter 3,

$$\left\{\lambda_{i,j}^k\left(\nu_k,\phi_k\right)\right\}_{i,j\leq N,k},\left\{\lambda_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j>N,k} \in S_V^N$$

such that the function $\varphi\left(x_1,x_2\right)$ defined as

$$\varphi\left(x_1,x_2\right) = \begin{cases} \left(u_{i,j}^k+\lambda_{i,j}^k\right)\left(\nu_k\left(x_1,x_2\right),\phi_k\left(x_1,x_2\right)\right) \; if \; \left(x_1,x_2\right)\in\Omega_{i,j}^k \; for \; j\leq N \\ \left(u_{i,j}^k+\lambda_{i,j}^k\right)\left(\xi\left(x_1,x_2\right),\eta\left(x_1,x_2\right)\right) \; if \; \left(x_1,x_2\right)\in\Omega_{i,j}^k \; for \; j> N \end{cases}$$

is a differentiable function of its arguments and $\varphi \in H_0^1\left(\Omega\right)$.

Moreover the estimate

$$\begin{aligned}(5.54)\quad & \sum_{k=1}^{p}\sum_{i=1}^{I_k}\left|\lambda_{i,1}^k\right|^2+\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\lambda_{i,j}^k\left(\nu_k,\phi_k\right)\right\|_{1,\widetilde{\Omega}_{i,j}^k}^2+\sum_{k=1}^{p}\sum_{j=N+1}^{J_k}\sum_{i=1}^{I_{k,j}}\left\|\lambda_{i,j}^k\left(\xi,\eta\right)\right\|_{1,S}^2 \\ \leq\; & C\left\{\left(\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\Gamma_l\cap\partial\Omega^k,\mu(\widetilde{\gamma}_s)<\infty}\left(\|u\|_{0,\widetilde{\gamma}_s}^2+\|u_{\nu_k}\|_{0,\widetilde{\gamma}_s}^2\right)\right)\right. \\ & +\sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k,\mu(\widetilde{\gamma}_s)<\infty}\left(\|[u]\|_{0,\widetilde{\gamma}_s}^2+\|[u_{\nu_k}]\|_{0,\widetilde{\gamma}_s}^2+\|[u_{\phi_k}]\|_{0,\widetilde{\gamma}_s}^2\right) \\ & +\sum_{k=1}^{p}\sum_{\gamma_s\subseteq B_p^k}\left(\|[u]\|_{0,\widetilde{\gamma}_s}^2+\|[u_{\nu_k}]\|_{0,\widetilde{\gamma}_s}^2+\|[u_{\phi_k}]\|_{0,\widetilde{\gamma}_s}^2\right) \\ & \left.+\sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\|[u]\|_{0,\gamma_s}^2+\left\|\left[\frac{\partial u}{\partial T}\right]\right\|_{0,\gamma_s}^2\right)+\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\left(\|u\|_{0,\gamma_s}^2+\left\|\frac{\partial u}{\partial T}\right\|_{0,\gamma_s}^2\right)\right\}\end{aligned}$$

holds.

We now explain the notation we have used in (5.54). Let $d\widehat{\sigma}$ denote an element of arc length in $\left(\nu_k,\phi_k\right)$ coordinates. Then

$$\|w\|_{0,\widetilde{\gamma}_s}^2 = \int_{\widetilde{\gamma}_s}\left|w\left(\nu_k,\phi_k\right)\right|^2 d\widehat{\sigma}.$$

Moreover if γ_s is given by $\gamma_s = \partial\Omega_m^{p+1} \bigcap \partial\Omega_n^{p+1}$ then

$$\left\| \left[\frac{\partial u}{\partial T} \right] \right\|_{0,\gamma_s}^2 = \int_{\gamma_s} \left(\frac{\partial u_m^{p+1}}{\partial T} - \frac{\partial u_n^{p+1}}{\partial T} \right)^2 ds.$$

Here $\frac{\partial}{\partial T}$ denotes the tangential derivative in (x_1, x_2) variables, i.e.

$$\frac{\partial u}{\partial T} = T^t \nabla_x u.$$

The other terms in the right hand side of (5.54) are similarly defined.

Now consider the bilinear form

$$\begin{aligned} B(\varphi, v) &= \int_\Omega \left(\sum_{r,s=1}^{2} a_{r,s}(x)\, \varphi_{x_s} v_{x_r} + \sum_{r=1}^{2} b_r(x)\, \varphi_{x_r} v + c\varphi v \right) dx \\ &= \sum_{k=1}^{p} \sum_{j=1}^{N} \sum_{i=1}^{I_k} B(\varphi, v)_{\Omega_{i,j}^k} + \sum_{l=1}^{L} B(\varphi, v)_{\Omega_l^{p+1}}. \end{aligned}$$

Here

$$B(\varphi, v)_\Delta = \int_\Delta \left(\sum_{r,s=1}^{2} a_{r,s}(x)\, \varphi_{x_s} v_{x_r} + \sum_{r=1}^{2} b_r(x)\, \varphi_{x_r} v + c\varphi v \right) dx$$

where Δ is a domain contained in Ω and $v \in H_0^1(\Omega)$.

Now

$$\begin{aligned} B(\varphi, v)_{\Omega_l^{p+1}} &= \int_{\Omega_l^{p+1}} \left(\sum_{r,s=1}^{2} a_{r,s}(x)\, \varphi_{x_s} v_{x_r} + \sum_{r=1}^{2} b_r(x)\, \varphi_{x_r} v + c\varphi v \right) dx \\ &= \int_{\Omega_l^{p+1}} \mathfrak{L}\varphi v dx + \int_{\partial\Omega_l^{p+1}} \left(\frac{\partial \varphi}{\partial N} \right)_A v ds. \end{aligned}$$

Similarly if $1 \leq j \leq N$ we have

$$B(\varphi, v)_{\Omega_{i,j}^k} = \int_{\tilde{\Omega}_{i,j}^k} \tilde{\mathfrak{L}}^k \varphi v d\tau_k d\theta_k + \int_{\partial\tilde{\Omega}_{i,j}^k} \left(\frac{\partial \varphi}{\partial n} \right)_{\tilde{A}^k} v d\sigma.$$

Moreover if $j = 1$

$$B(\varphi, v)_{\Omega_{i,1}^k} = \int_{\tilde{\Omega}_{i,1}^k} c\varphi v e^{2\tau_k} d\tau_k d\theta_k + \int_{\partial\tilde{\Omega}_{i,1}^k} \left(\frac{\partial \varphi}{\partial n} \right)_{\tilde{A}^k} v d\sigma$$

since φ is a constant on $\widetilde{\Omega}^k_{i,1}$.

Finally if $j = N+1$ we obtain

$$B(\varphi, v)_{\Omega^k_{i,N+1}} = \int_{\Omega^k_{i,N+1}} \mathfrak{L}\varphi v dx + \int_{\widetilde{B}^k_\rho} \left(\frac{\partial\varphi}{\partial n}\right)_{\widetilde{A}^k} v d\sigma + \int_{\partial\Omega^k_{i,N+1}\setminus B^k_\rho} \left(\frac{\partial\varphi}{\partial N}\right)_A vds.$$

For by (5.39b)

$$\rho\left(\frac{\partial\varphi}{\partial N}\right)_A (P) = \left(\frac{\partial\varphi}{\partial n}\right)_{\widetilde{A}^k} \left(\widetilde{P}\right)$$

and $ds = \rho d\sigma$. Here P is any point on the circular arc B^k_ρ and $\widetilde{P}$ is it's image in (τ_k, θ_k) coordinates. Now

$$
\begin{aligned}
(5.55)\quad B(\varphi, v) &= \sum_{k=1}^{p}\sum_{j=1}^{N}\sum_{i=1}^{I_k} B(\varphi, v)_{\Omega^k_{i,j}} + \sum_{l=1}^{L} B(\varphi, v)_{\Omega^{p+1}_l} \\
&= \sum_{k=1}^{p}\sum_{j=1}^{N}\sum_{i=1}^{I_k} B\left(u^k_{i,j}, v\right)_{\Omega^k_{i,j}} + \sum_{l=1}^{L} B\left(u^{p+1}_l, v\right)_{\Omega^{p+1}_l} \\
&+ \left(\sum_{k=1}^{p}\sum_{j=1}^{N}\sum_{i=1}^{I_k} B\left(\lambda^k_{i,j}, v\right)_{\Omega^k_{i,j}} + \sum_{l=1}^{L} B\left(\lambda^{p+1}_l, v\right)_{\Omega^{p+1}_l}\right) \\
&= \sum_{k=1}^{p}\sum_{j=1}^{N}\sum_{i=1}^{I_k} \int_{\widetilde{\Omega}^k_{i,j}} \widetilde{\mathfrak{L}}^k u^k_{i,j} v d\tau_k d\theta_k + \sum_{l=1}^{L} \int_{\Omega^{p+1}_l} \mathfrak{L} u^{p+1}_l v dx_1 dx_2 \\
&+ \sum_{k=1}^{p} \sum_{\gamma_s \subseteq \Omega^k, \mu(\widetilde{\gamma}_s)<\infty} \int_{\widetilde{\gamma}_s} \left[\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right] v d\sigma \\
&+ \sum_{\gamma_s \subseteq \Omega^{p+1}} \int_{\gamma_s} \left[\left(\frac{\partial u}{\partial N}\right)_A\right] vds + \sum_{k=1}^{p} \sum_{\gamma_s \subseteq B^k_\rho} \int_{\widetilde{\gamma}_s} \left[\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right] v d\sigma \\
&+ \sum_{l \in \mathcal{N}} \sum_{k=l-1}^{l} \sum_{\gamma_s \subseteq \Gamma_l \cap \partial\Omega^k, \mu(\widetilde{\gamma}_s)<\infty} \int_{\widetilde{\gamma}_s} \left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k} v d\sigma \\
&+ \sum_{l \in \mathcal{N}} \sum_{r=1}^{L} \sum_{\gamma_s \subseteq \partial\Omega^{p+1}_r \cap \Gamma_l} \int_{\gamma_s} \left(\frac{\partial u}{\partial N}\right)_A vds \\
&+ \left(\sum_{k=1}^{p}\sum_{i=1}^{I_k} B\left(\lambda^k_{i,1}, v\right)_{\Omega^k_{i,1}} + \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k} B\left(\lambda^k_{i,j}, v\right)_{\Omega^k_{i,j}} \right. \\
&+ \left. \sum_{l=1}^{L} B\left(\lambda^{p+1}_l, v\right)_{\Omega^{p+1}_l}\right).
\end{aligned}
$$

Now

$$\int_{\widetilde{\Omega}_{i,1}^k} \widetilde{\mathfrak{L}}^k \lambda_{i,1}^k v d\tau_k d\theta_k = \int_{\widetilde{\Omega}_{i,1}^k} c\lambda_{i,1}^k v e^{2\tau_k} d\tau_k d\theta_k.$$

Here

$$\lambda_{i,1}^k = \left\{ \begin{array}{cc} -u_{i,1}^k & \text{if } \Gamma_k \text{ or } \Gamma_{k+1} \subseteq \Gamma^{[0]}, \text{ and} \\ 0 & \text{otherwise} \end{array} \right\}.$$

Now $c_k = c(A_k)$, a constant, and $c(x_1, x_2)$ is an analytic function of x_1 and x_2. Hence

$$\left| \int_{\widetilde{\Omega}_{i,1}^k} \widetilde{\mathfrak{L}}^k \lambda_{i,1}^k v d\tau_k d\theta_k \right| \leq 2c_k \left(\int \left| \lambda_{i,1}^k \right|^2 e^{2\tau_k} d\tau_k d\theta_k \right)^{1/2} \times \left(\int v^2 e^{2\tau_k} d\tau_k d\theta_k \right)^{1/2}$$

for N large enough. And so we obtain

$$\left| \int_{\widetilde{\Omega}_{i,1}^k} \widetilde{\mathfrak{L}}^k \lambda_{i,1}^k v d\tau_k d\theta_k \right| \leq \varepsilon \left| \lambda_{i,1}^k \right| \left\| v(x_1, x_2) \right\|_{0,\Omega_{i,1}^k}.$$

where ε is exponentially small in N. Now let $2 \leq j \leq N$. Then

$$\left| \int_{\widetilde{\Omega}_{i,j}^k} \widetilde{\mathfrak{L}}^k u_{i,j}^k v d\tau_k d\theta_k \right| \leq \left\| \widetilde{\mathfrak{L}}^k u_{i,j}^k (\tau_k, \theta_k) \right\|_{0,\widetilde{\Omega}_{i,j}^k} \left\| v(\tau_k, \theta_k) \right\|_{0,\widetilde{\Omega}_{i,j}^k}.$$

Finally

$$\left| \int_{\Omega_l^{p+1}} \left(\mathfrak{L} u_l^{p+1} \right) v dx \right| \leq \left\| \mathfrak{L} u_l^{p+1} (x_1, x_2) \right\|_{0,\Omega_l^{p+1}} \left\| v(x_1, x_2) \right\|_{0,\Omega_l^{p+1}}.$$

Now

$$\sum_{k=1}^{p} \sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| v(\nu_k, \phi_k) \right\|_{0,\widetilde{\Omega}_{i,j}^k}^2 \leq CN^2 \left\| v(x_1, x_2) \right\|_{1,\Omega}^2.$$

Hence

$$\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\|v(\nu_k,\phi_k)\|^2_{1,\widehat{\Omega}^k_{i,j}} \le CN^2\|v(x_1,x_2)\|^2_{1,\Omega}. \tag{5.56}$$

Now using the trace theorem for Sobolev spaces we obtain

$$\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\|v\|^2_{0,\partial\widehat{\Omega}^k_{i,j}} \le CN^2\|v(x_1,x_2)\|^2_{1,\Omega}.$$

And so we can conclude that

$$\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\int_{\partial\widetilde{\Omega}^k_{i,j}} v^2 d\sigma \le CN^2\|v(x_1,x_2)\|^2_{1,\Omega}. \tag{5.57}$$

Using the Cauchy Schwartz inequality in (5.55) and using (5.56) and (5.57) we can conclude that

$$\begin{aligned}
|B(\varphi,v)|^2 \le\ & K\Bigg\{\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\widetilde{\mathfrak{L}}^k u^k_{i,j}(\tau_k,\theta_k)\right\|^2_{0,\widetilde{\Omega}^k_{i,j}} + \sum_{k=1}^{p}\sum_{i=1}^{I_k}\varepsilon\left|u^k_{i,1}\right|^2 \\
&+ \sum_{k=1}^{p}\left(\sum_{\gamma_s\subseteq\Omega^k}\int_{\widetilde{\gamma}_s}\left[\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right]^2 d\sigma + \sum_{\gamma_s\subseteq B^k_p}\int_{\widetilde{\gamma}_s}\left[\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}\right]^2 d\sigma\right) \\
&+ \sum_{l=1}^{L}\int_{\Omega^{p+1}_l}\int\left|\mathfrak{L}u^{p+1}_l(x_1,x_2)\right|^2 dx_1dx_2 + \sum_{\gamma_s\subseteq\Omega^{p+1}}\int_{\gamma_s}\left[\left(\frac{\partial u}{\partial N}\right)_A\right]^2 ds \\
&+ \sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\Gamma_l\cap\partial\Omega^{p+1}}\int_{\gamma_s}\left(\frac{\partial u}{\partial N}\right)^2_A ds + \varepsilon\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left|\lambda^k_{i,1}\right|^2 \\
&+ \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\lambda^k_{i,j}(\tau_k,\theta_k)\right\|^2_{1,\widetilde{\Omega}^k_{i,j}} + \sum_{l=1}^{L}\left\|\lambda^{p+1}_l(x,y)\right\|^2_{1,\Omega^{p+1}_l}\Bigg\} \\
&\bullet\ \Bigg\{\sum_{k=1}^{p}\sum_{i=1}^{I_k}\|v(x_1,x_2)\|^2_{0,\Omega^k_{i,1}} + \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\|v(\tau_k,\theta_k)\|^2_{1,\widetilde{\Omega}^k_{i,j}} \\
&+ \sum_{l=1}^{L}\left\|v^{p+1}_l(x,y)\right\|^2_{1,\Omega^{p+1}_l} + \sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\int_{\partial\widetilde{\Omega}^k_{i,j}} v^2 d\sigma \\
&+ \sum_{\gamma_s\subseteq\Omega^{p+1}}\int_{\gamma_s} v^2 ds + \sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\cap\Gamma_l}\int_{\gamma_s} v^2 ds\Bigg\}.
\end{aligned}$$

Now $v \in H_0^1(\Omega)$ and $\mathfrak{L}$ satisfies the inf-sup conditions (5.15a - 5.15b). Hence using (5.16), (5.54) and (5.57) we obtain

$$
\begin{aligned}
\|\varphi\|_{1,\Omega}^2 \leq\ & CN^2 \Bigg\{ \sum_{k=1}^{p} \sum_{j=2}^{N} \sum_{i=1}^{I_k} \left\| \widetilde{\mathfrak{L}}^k u_{i,j}^k (\tau_k, \theta_k) \right\|_{0,\widetilde{\Omega}_{i,j}^k}^2 \\
& + \sum_{k=1}^{p} \Bigg(\sum_{\gamma_s \subseteq \Omega^k} \left(\|[u]\|_{0,\widetilde{\gamma}_s}^2 + \|[u_{\nu_k}]\|_{0,\widetilde{\gamma}_s}^2 + \|[u_{\phi_k}]\|_{0,\widetilde{\gamma}_s}^2 \right) \\
& + \sum_{l \in \mathcal{N}} \sum_{k=l-1}^{l} \sum_{\gamma_s \subseteq \partial\Omega^k \cap \Gamma_l} \int_{\widetilde{\gamma}_s} \left(\frac{\partial u}{\partial n} \right)_{\widetilde{A}^k}^2 d\sigma \\
& + \sum_{l \in \mathcal{D}} \sum_{k=l-1}^{l} \sum_{\gamma_s \subseteq \Gamma_l \cap \partial\Omega^k, \mu(\widetilde{\gamma}_s) < \infty} \left(\|u\|_{0,\widetilde{\gamma}_s}^2 + \|u_{\nu_k}\|_{0,\widetilde{\gamma}_s}^2 \right) \Bigg) \\
& + \sum_{k=1}^{p} \sum_{\gamma_s \subseteq B_p^k} \left(\|[u]\|_{0,\widehat{\gamma}_s}^2 + \|[u_{\nu_k}]\|_{0,\widehat{\gamma}_s}^2 + \|[u_{\phi_k}]\|_{0,\widehat{\gamma}_s}^2 \right) \\
& + \sum_{l=1}^{L} \int_{\Omega_l^{p+1}} \int \left| \mathfrak{L}_l^{p+1} u_l^{p+1} (x_1, x_2) \right|^2 dx_1 dx_2 \\
& + \sum_{\gamma_s \subseteq \Omega^{p+1}} \left(\|[u]\|_{0,\gamma_s}^2 + \|[u_{x_1}]\|_{0,\gamma_s}^2 + \|[u_{x_2}]\|_{0,\gamma_s}^2 \right) \\
& + \sum_{l \in \mathcal{D}} \sum_{\gamma_s \subseteq \partial\Omega^{p+1} \cap \Gamma_l} \left(\|u\|_{0,\gamma_s}^2 + \left\| \frac{\partial u}{\partial T} \right\|_{0,\gamma_s}^2 \right) \\
& + \sum_{l \in \mathcal{N}} \sum_{\gamma_s \subseteq \partial\Omega^{p+1} \cap \Gamma_l} \int \left(\frac{\partial u}{\partial N} \right)_A^2 ds + \varepsilon \sum_{k=1}^{p} \sum_{i=1}^{I_k} \left(\left| u_{i,1}^k \right|^2 + \left| \lambda_{i,1}^k \right|^2 \right) \Bigg\}.
\end{aligned}
$$

Here ε is exponentially small in N.

Using (5.54) and (5.56) once more we obtain the result.□

We now define differential operators $\left(\mathfrak{L}_{i,j}^k\right)^a$ which are second order differential operators with polynomial coefficients in ν_k and ϕ_k of degree $(N-1)$ such that these coefficients are exponentially close approximation to the coefficients of $\left(\mathfrak{L}_{i,j}^k\right)$ as has been described in the discussion leading to the inequality (2.69). In the same way we define the differential operator $\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^a$ to be a first order differential operator with polynomial coefficients in ν_k and ϕ_k such that these coefficients are exponentially close approximations to the coefficients of $\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}$ as has been done in (2.55a - 2.55b). The

other approximations are similarly defined.

From the above it is easy to conclude that

$$\begin{aligned}(5.58)\qquad & \sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\left|u_{i,1}^{k}\right|^2+\sum_{j=2}^{N}\left\|u_{i,j}^{k}\left(\nu_k,\phi_k\right)\right\|_{1,\widehat{\Omega}_{i,j}^{k}}^{2}\right)+\sum_{l=1}^{L}\left\|u_l^{p+1}\left(\xi,\eta\right)\right\|_{1,S}^{2}\\ \leq\ & CN^4\left(\mathcal{I}\right)\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}\ =\ & \Bigg\{\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\left(\mathfrak{L}_{i,j}^{k}\right)^{a}u_{i,j}^{k}\left(\nu_k,\phi_k\right)\right\|_{0,\widehat{\Omega}_{i,j}^{k}}^{2}\\ & +\sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k}\left(\left\|[u]\right\|_{0,\widehat{\gamma}_s}^{2}+\left\|[u_{\nu_k}]\right\|_{0,\widehat{\gamma}_s}^{2}+\left\|[u_{\phi_k}]\right\|_{0,\widehat{\gamma}_s}^{2}\right)\\ & +\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty}\left(\left\|u\right\|_{0,\widehat{\gamma}_s}^{2}+\left\|u_{\nu_k}\right\|_{0,\widehat{\gamma}_s}^{2}\right)\\ & +\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty}\left\|\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^{a}\right\|_{0,\widehat{\gamma}_s}^{2}\\ & +\sum_{k=1}^{p}\sum_{\gamma_s\subseteq B_\rho^k}\left(\left\|[u]\right\|_{0,\widehat{\gamma}_s}^{2}+\left\|[u_{\nu_k}]\right\|_{0,\widehat{\gamma}_s}^{2}+\left\|[u_{\phi_k}]\right\|_{0,\widehat{\gamma}_s}^{2}\right)\\ & +\sum_{l=1}^{L}\left\|\left(\mathfrak{L}_l^{p+1}\right)^{a}u_l^{p+1}\left(\xi,\eta\right)\right\|_{0,S}^{2}+\sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\left\|[u]\right\|_{0,\gamma_s}^{2}+\left\|[u_{x_1}]^{a}\right\|_{0,\gamma_s}^{2}+\left\|[u_{x_2}]^{a}\right\|_{0,\gamma_s}^{2}\right)\\ & +\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left(\left\|u\right\|_{0,\gamma_s}^{2}+\left\|\left(\frac{\partial u}{\partial T}\right)^{a}\right\|_{0,\gamma_s}^{2}\right)+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left\|\left(\frac{\partial u}{\partial N}\right)_{A}^{a}\right\|_{0,\gamma_s}^{2}\Bigg\}.\end{aligned}$$

Adding a weighted combination of (5.51), (5.52) and (5.58) and using the techniques and results of previous chapters it is not difficult to prove Theorem 5.2 which is stated below.

Theorem 5.2 *For N large enough the estimate*

$$\begin{aligned}(5.59)\qquad & \sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\left|u_{i,1}^{k}\right|^2+\sum_{j=2}^{N}\left\|u_{i,j}^{k}\left(\nu_k,\phi_k\right)\right\|_{2,\widehat{\Omega}_{i,j}^{k}}^{2}\right)+\sum_{l=1}^{L}\left\|u_l^{p+1}\left(\xi,\eta\right)\right\|_{2,S}^{2}\\ \leq\ & CN^4\mathcal{V}^N\left(\left\{u_{i,j}^{k}\left(\nu_k,\phi_k\right)\right\}_{i,j,k},\left\{u_{i,j}^{k}\left(\xi,\eta\right)\right\}_{i,j,k}\right)\end{aligned}$$

other approximations are similarly defined.

From the above it is easy to conclude that

$$\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\left|u_{i,1}^k\right|^2+\sum_{j=2}^{N}\left\|u_{i,j}^k\left(\nu_k,\phi_k\right)\right\|_{1,\widehat{\Omega}_{i,j}^k}^2\right)+\sum_{l=1}^{L}\left\|u_l^{p+1}\left(\xi,\eta\right)\right\|_{1,S}^2 \tag{5.58}$$
$$\leq\ CN^4\left(\mathcal{I}\right)$$

where

$$\begin{aligned}\mathcal{I}\ =\ &\left\{\sum_{k=1}^{p}\sum_{j=2}^{N}\sum_{i=1}^{I_k}\left\|\left(\mathfrak{L}_{i,j}^k\right)^a u_{i,j}^k\left(\nu_k,\phi_k\right)\right\|_{0,\widehat{\Omega}_{i,j}^k}^2\right.\\
&+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k}\left(\|[u]\|_{0,\widetilde{\gamma}_s}^2+\|[u_{\nu_k}]\|_{0,\widetilde{\gamma}_s}^2+\|[u_{\phi_k}]\|_{0,\widetilde{\gamma}_s}^2\right)\\
&+\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widetilde{\gamma}_s)<\infty}\left(\|u\|_{0,\widetilde{\gamma}_s}^2+\|u_{\nu_k}\|_{0,\widetilde{\gamma}_s}^2\right)\\
&+\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widetilde{\gamma}_s)<\infty}\left\|\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^a\right\|_{0,\widetilde{\gamma}_s}^2\\
&+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq B_p^k}\left(\|[u]\|_{0,\widetilde{\gamma}_s}^2+\|[u_{\nu_k}]\|_{0,\widetilde{\gamma}_s}^2+\|[u_{\phi_k}]\|_{0,\widetilde{\gamma}_s}^2\right)\\
&+\sum_{l=1}^{L}\left\|\left(\mathfrak{L}_l^{p+1}\right)^a u_l^{p+1}\left(\xi,\eta\right)\right\|_{0,S}^2+\sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\|[u]\|_{0,\gamma_s}^2+\|[u_{x_1}]^a\|_{0,\gamma_s}^2+\|[u_{x_2}]^a\|_{0,\gamma_s}^2\right)\\
&\left.+\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left(\|u\|_{0,\gamma_s}^2+\left\|\left(\frac{\partial u}{\partial T}\right)^a\right\|_{0,\gamma_s}^2\right)+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left\|\left(\frac{\partial u}{\partial N}\right)_A^a\right\|_{0,\gamma_s}^2\right\}.\end{aligned}$$

Adding a weighted combination of (5.51), (5.52) and (5.58) and using the techniques and results of previous chapters it is not difficult to prove Theorem 5.2 which is stated below.

Theorem 5.2 *For N large enough the estimate*

$$\sum_{k=1}^{p}\sum_{i=1}^{I_k}\left(\left|u_{i,1}^k\right|^2+\sum_{j=2}^{N}\left\|u_{i,j}^k\left(\nu_k,\phi_k\right)\right\|_{2,\widehat{\Omega}_{i,j}^k}^2\right)+\sum_{l=1}^{L}\left\|u_l^{p+1}\left(\xi,\eta\right)\right\|_{2,S}^2 \tag{5.59}$$
$$\leq\ CN^4\mathcal{V}^N\left(\left\{u_{i,j}^k\left(\nu_k,\phi_k\right)\right\}_{i,j,k},\left\{u_{i,j}^k\left(\xi,\eta\right)\right\}_{i,j,k}\right)$$

where

$$\mathcal{V}^N\left(\left\{u_{i,j}^k(\nu_k,\phi_k)\right\}_{i,j,k},\left\{u_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right) \tag{5.60}$$

$$= \left\{\sum_{k=1}^{p}\sum_{j=2}^{N}\left\|\left(\mathfrak{L}_{i,j}^k\right)^a u_{i,j}^k(\nu_k,\phi_k)\right\|_{0,\widehat{\Omega}_{i,j}^k}^2\right.$$

$$+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq\Omega^k}\left(\left\|[u]\right\|_{0,\widehat{\gamma}_s}^2+\left\|[u_{\nu_k}]\right\|_{1/2,\widehat{\gamma}_s}^2+\left\|[u_{\phi_k}]\right\|_{1/2,\widehat{\gamma}_s}^2\right)$$

$$+\sum_{l\in\mathcal{D}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty}\left(\left\|u\right\|_{0,\widehat{\gamma}_s}^2+\left\|u_{\nu_k}\right\|_{1/2,\widehat{\gamma}_s}^2\right)$$

$$+\sum_{l\in\mathcal{N}}\sum_{k=l-1}^{l}\sum_{\gamma_s\subseteq\partial\Omega^k\bigcap\Gamma_l,\mu(\widehat{\gamma}_s)<\infty}\left\|\left(\frac{\partial u}{\partial n}\right)_{\widetilde{A}^k}^a\right\|_{1/2,\widehat{\gamma}_s}^2$$

$$+\sum_{k=1}^{p}\sum_{\gamma_s\subseteq B_p^k}\left(\left\|[u]\right\|_{0,\widehat{\gamma}_s}^2+\left\|[u_{\nu_k}]\right\|_{1/2,\widehat{\gamma}_s}^2+\left\|[u_{\phi_k}]\right\|_{1/2,\widehat{\gamma}_s}^2\right)$$

$$+\sum_{l=1}^{L}\left\|\left(\mathfrak{L}_l^{p+1}\right)^a u_l^{p+1}(\xi,\eta)\right\|_{0,S}^2$$

$$+\sum_{\gamma_s\subseteq\Omega^{p+1}}\left(\left\|[u]\right\|_{0,\gamma_s}^2+\left\|[u_{x_1}]^a\right\|_{1/2,\gamma_s}^2+\left\|[u_{x_2}]^a\right\|_{1/2,\gamma_s}^2\right)$$

$$+\sum_{l\in\mathcal{D}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left(\left\|u\right\|_{0,\gamma_s}^2+\left\|\left(\frac{\partial u}{\partial T}\right)^a\right\|_{1/2,\gamma_s}^2\right)$$

$$\left.+\sum_{l\in\mathcal{N}}\sum_{\gamma_s\subseteq\partial\Omega^{p+1}\bigcap\Gamma_l}\left\|\left(\frac{\partial u}{\partial N}\right)_A^a\right\|_{1/2,\gamma_s}^2\right\}$$

holds for all $\left\{\left\{u_{i,j}^k(\nu_k,\phi_k)\right\}_{i,j,k},\left\{u_{i,j}^k(\xi,\eta)\right\}_{i,j,k}\right\}\in S_V^N$.□

5.4 Conclusion

The numerical scheme is formulated in exactly the same way as in the earlier chapters. Construction of preconditioners and error estimates also proceed in just the same way.

With this we have given a complete description and analysis of a *least-squares* approach to solving elliptic boundary value problems in polygonal domains.

The method is a *vertex based method* and the spectral element functions are con-

tinuous *only* at the vertices. As a result the *Schur complement matrix* has small dimension and an accurate inverse can be computed. Hence the numerical scheme has a computational complexity which is less then that of FEM.

Moreover, the construction of a preconditioner for the Schur complement matrix is very simple unlike the case for FEM. In fact, for problems in 3 dimensions the construction of preconditioners for the Schur complement matrix becomes very complex for FEM [40].

The ideas in these chapters, though they deal with problems in two dimensions, generalize to 3 dimensions. We intend to study these problems in 3 dimensions both theoretically and computationally in future work.

Bibliography

[1] M. Abramowitz and I.A. Stegun, eds (1965): Handbook of Mathematical Functions, Dover.

[2] R.A. Adams (1975): Sobolev Spaces, Academic Press.

[3] M. Ainsworth (1996): A Preconditioner Based on Domain Decomposition for *h-p* Finite Element Approximation on Quasi-Uniform Meshes, SIAM J. Num. Anal., 33, 1358-1376.

[4] O. Axelsson (1994): Iterative Solution Methods, Cambridge Univ. Press.

[5] I. Babuska, A. Craig, J. Mandel and J. Pitkaranta (1991): Efficient Preconditioning for the p version finite element method in two dimensions, SIAM J. Num. Anal., 28, 624-661.

[6] I. Babuska and M.R. Dorr (1981): Error Estimates for the Combined h and p Versions of the Finite Element Method, Num. Math., 37, 257-277.

[7] I. Babuska and B.Q. Guo (1988): Regularity of the Solution of Elliptic Problems with Piecewise Analytic Data, Part - I, SIAM J. Math. Anal., 19, 172-203.

[8] I. Babuska and B.Q. Guo (1988): The h-p Version of the Finite Element Method on Domains with Curved Boundaries, SIAM J. Num. Anal., 25, 837-861.

[9] I. Babuska, R.B. Kellogg and J. Pitkaranta (1979): Direct and Inverse Error Estimates for Finite Elements with Mesh Refinements, Num. Math., 33, 447-471.

[10] I. Babuska and H-S Oh (1990): The p version of finite element method for domains with corners and for infinite domains. Numer. Mathe. PDEs, 6, 371.

[11] I. Babuska and M. Suri (1994): The p and h-p versions of the finite element method, Basic Principles and Properties, SIAM Review, 36, 578-632.

[12] J.H. Bramble, J.E. Pasciak and A.H. Schatz (1986): The Construction of Preconditioners for Elliptic Problems by Substructuring I, Math. Comp. 47, 103-134.

[13] S.C. Brenner and L.R. Scott (1994): The Mathematical Theory of Finite Element Methods, Springer Verlag.

[14] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang (1988): Spectral Methods in Fluid Dynamics, Series in Computational Physics, Springer Verlag.

[15] P.G. Ciarlet (1978): The Finite Element Method for Elliptic Problems, North Holland, Amsterdam.

[16] P.J. Davis and P. Rabinowitz (1984): Methods of Numerical Integration, 2nd ed., Academic Press.

[17] P. Dutt (1990): Spectral Methods for Initial Boundary Value Problems - An Alternative Approach, SIAM J. Num. Anal., 27, 885-903.

[18] P. Dutt and S. Joshi: Spectral Methods for Hyperbolic Initial Boundary Value Problems on Parallel Computers, to appear in Jour. Comp. and Appl. Math.

[19] P.K. Dutt and A.K. Singh (1994): The Galerkin-Collocation Method for Hyperbolic Initial Boundary Value Problems, Jour. Comp. Phys., 112, 211-225.

[20] G.H. Golub and J.M. Ortega (1993): Scientific Computing, an Introduction with Parallel Computing, Academic Press.

[21] G.H. Golub and C.F. Van Loan (1996): Matrix Computations, 3rd ed., Johns Hopkins Univ. Press.

[22] D. Gottlieb and S.A. Orszag (1977): Numerical Analysis of Spectral Methods: Theory and Applications, SIAM.

[23] P. Grisvard (1985): Elliptic Problems in Non-smooth Domains, Pitman Advanced Publishing Program.

[24] W. Gropp, E. Lusk and A. Skjellum (1994): Using MPI, Portable Parallel Programming with the Message-Passing Interface, MIT Press.

[25] W. Gropp, E. Lusk and R. Thakur (1999): Using MPI-2, Advanced Features of the Message-Passing Interface, MIT Press.

[26] B. Guo and W. Cao (1996): A preconditioner for the *h-p* version of the Finite Element Method in two dimensions, Num. Math., 75, 59-77.

[27] Hardy, Littlewood and Polya (1959): Inequalities, Cambridge Univ. Press.

[28] G. Karniadakis and Sherwin J. Spencer (1999): Spectral/*hp* Element Methods for CFD, Oxford University Press.

[29] S. Kesavan (1989): Topics in Functional Analysis and Applications, Wiley Eastern Limited.

[30] V.A. Kondratiev (1970): The Smoothness of a Solution of Dirichlet's Problem for Second Order Elliptic Equations in a Region with a Piecewise Smooth Boundary, Differential' nye Uraneniya, 6(10), 1831-1843 (and Differential Equations, 6, 1392-1401).

[31] J.L. Lions and E. Magenes (1972): Non-homogeneous Boundary Value Problems and Applications -I, Springer Verlag, Berlin, Heidelberg, New York.

[32] Message Passing Interface Forum (1994): MPI: A message-passing interface standard.

[33] Message Passing Interface Forum (1997): MPI-2: Extensions to the message-passing interface.

[34] J.J. Modi (1988): Parallel Algorithms and Matrix Computation, Clarendon Press, Oxford.

[35] J.T. Oden (1979): Applied Functional Analysis, Prentice-Hall, Inc., New Jersey.

[36] J.T. Oden, A. Patra and Y.S. Feng (1994): Parallel Domain Decomposition Solver for Adaptive *h-p* Finite Element Methods. TICAM Report 94-11.

[37] J.M. Ortega (1988): Introduction to Parallel and Vector Solution of Linear Systems, Plenum Press, New York.

[38] A. Patera (1984): A Spectral Element Method for fluid dynamics: Laminar flow in a channel expansion, Jour. Comp. Phys., 54, 468-488.

[39] D. Pathria and G.E. Karniadakis (1995): Spectral Element Methods for Elliptic Problems in Non-smooth Domains, Jour. Comp. Phy., 122, 83-95.

[40] L. Pavarino and O. Widlund (1996): A polylogarithmic bound for an iterative substructuring method for spectral elements in three dimensions, SIAM J. Num. Anal., 33, 1303-1335.

[41] M.J. Quinn (1994): Parallel Computing, Theory and Practice, McGraw-Hill, Inc.

[42] Ch. Schwab (1998) : *p*- and *hp*- Finite Element Methods, Clarendon Press, Oxford.

[43] W.G. Strang and G.J. Fix (1973): An Analysis of the Finite Element of Method, Prentice-Hall, Englewood Cliffs, New Jersey.

[44] R.S. Varga (1962): Matrix Iterative Analysis, Prentice-Hall, Englewood Cliffs, New Jersey.

[45] R.G. Voigt, D. Gottlieb and M.Y. Hussaini (1984): Spectral Methods for Partial Differential Equations, SIAM.